DE June 2011.

1. (i) $$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 13\cos 2t$$

Auxiliary Equation for LHS

$\lambda^2 + 4\lambda + 3 = 0$

$(\lambda+1)(\lambda+3) = 0$

$\therefore \lambda = -1$ or $\lambda = -3$

$\therefore$ Complementary Function

$y = Ae^{-t} + Be^{-3t}$

Particular Integral is based on LHS

$y = C\sin 2t + D\cos 2t$

$\therefore \dot{y} = 2C\cos 2t - 2D\sin 2t$

$\ddot{y} = -4C\sin 2t - 4D\cos 2t$

Substituting into D.E. gives

$-4C\sin 2t - 4D\cos 2t$

$+4(2C\cos 2t - 2D\sin 2t)$

$+3(C\sin 2t + D\cos 2t) = 13\cos 2t$

cf: $\cos 2t$ $\quad -4D + 8C + 3D = 13$

$8C - D = 13$

cf: $\sin 2t$ $\quad -4C - 8D + 3C = 0$

$-C - 8D = 0$ $\quad \frac{\times 8}{\text{add}}$

$-65D = 13$

$\therefore D = -\frac{1}{5}$

and $C = \frac{8}{5}$

So Particular Integral

$y = \frac{8}{5}\sin 2t - \frac{1}{5}\cos 2t$

Since GS = CF + PI

then

$y = Ae^{-t} + Be^{-3t} + \frac{8}{5}\sin 2t - \frac{1}{5}\cos 2t$

(ii) when $t=0$, $y=0$, $\dot{y}=0$

$0 = A + B - \frac{1}{5}$

Also $\dot{y} = -Ae^{-t} - 3Be^{-3t} + \frac{16}{5}\cos 2t + \frac{2}{5}\sin 2t$

$\therefore 0 = -A - 3B + \frac{16}{5}$

$$\left.\begin{aligned} \therefore 5A + 5B &= 1 \\ 5A + 15B &= 16 \end{aligned}\right\} \quad 10B = 15$$

$B = \frac{3}{2}$, $A = -\frac{13}{10}$

The Particular Solution is $y = -\frac{13}{10}e^{-t} + \frac{3}{2}e^{-3t} + \frac{8}{5}\sin 2t - \frac{1}{5}\cos 2t.$

(iii) $$\frac{d^3z}{dt^3} + 4\frac{d^2z}{dt^2} + 3\frac{dz}{dt} = -26\sin 2t$$

Let $z = y + c \Rightarrow \dot{z} = \dot{y}$ and $\ddot{z} = \ddot{y}$ and $\dddot{z} = \dddot{y}$

$\therefore \dddot{y} + 4\ddot{y} + 3\dot{y} = -26\sin 2t$

from (i) we can d wrt 't' to give $\dddot{y} + 4\ddot{y} + 3\dot{y} = -26\sin 2t$

both equations are the same so $z = y + c$ must be a solution

where y is GS from part (i)

(iv) when $t=0$, $z=2$, $\dot{z}=0$, and $\ddot{z}=13$ (3 unknowns)

$$z = Ae^{-t} + Be^{-3t} + \frac{8}{5}\sin 2t - \frac{1}{5}\cos 2t + c$$

$$\Rightarrow A + B - \frac{1}{5} + c = 2 \quad ①$$

$$\dot{z} = -Ae^{-t} - 3Be^{-3t} + \frac{16}{5}\cos 2t + \frac{2}{5}\sin 2t$$

$$\Rightarrow -A - 3B + \frac{16}{5} = 0 \quad ②$$

$$\ddot{z} = Ae^{-t} + 9Be^{-3t} - \frac{32}{5}\sin 2t + \frac{4}{5}\cos 2t$$

$$\Rightarrow A + 9B + \frac{4}{5} = 13 \quad ③$$

② + ③ $\Rightarrow$ $6B + 4 = 13$ $\Rightarrow$ $B = \frac{3}{2}$, $A = -\frac{13}{10}$, $c = 2$

$$\therefore \quad z = -\frac{13}{10}e^{-t} + \frac{3}{2}e^{-3t} + \frac{8}{5}\sin 2t - \frac{1}{5}\cos 2t + 2$$

2(i) $\dfrac{dy}{dx} - \dfrac{2y}{x} = \sqrt{x} \quad (x>0)$

$\dfrac{dy}{dx} + P(x)\,y = Q(x)$ $\quad P(x) = -\dfrac{2}{x}$

Integrating Factor $= e^{\int P(x)\,dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$

$$\therefore \frac{d}{dx}\left(x^{-2}y\right) = x^{-2}.x^{1/2} = x^{-3/2}$$

$$\therefore x^{-2}y = \int x^{-3/2}\,dx$$

$$\therefore \frac{y}{x^2} = \frac{x^{-1/2}}{-1/2} + c$$

$$\therefore y = -2x^{3/2} - 2cx^2$$

$$\therefore y = -2x^{3/2} + Ax^2$$

(ii) when $x=1$, $y=0$ $\quad \therefore 0 = -2 + A \Rightarrow A = 2$

$$\therefore y = -2x^{3/2} + 2x^2$$

(iii) stationary points when $\dfrac{dy}{dx} = 0$ $\qquad \dfrac{dy}{dx} = -3x^{1/2} + 4x$

If $4x - 3x^{1/2} = 0$

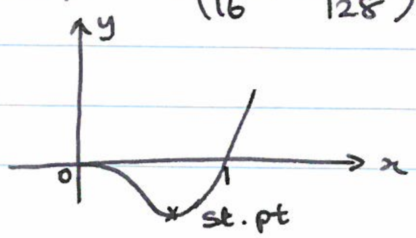
$x^{1/2}(4x^{1/2} - 3) = 0$ either $x^{1/2} = 0$ $(x>0)$ not true

or $x^{1/2} = 3/4 \Rightarrow x = \frac{9}{16}$

If $x = \frac{9}{16}$ then $y = \frac{-27}{128}$ so st. pt is $\left(\frac{9}{16}, \frac{-27}{128}\right)$

If $x \to 0$ then $y \to 0$

If $x \to 0$ then $\dfrac{dy}{dx} \to 0$



2(b) $\frac{dy}{dx} = \sqrt{x^2+y^2}$ using tangent fields.

(i) isocline for which $\frac{dy}{dx} = 1$ is the curve joining all points with $\frac{dy}{dx} = 1$, ie

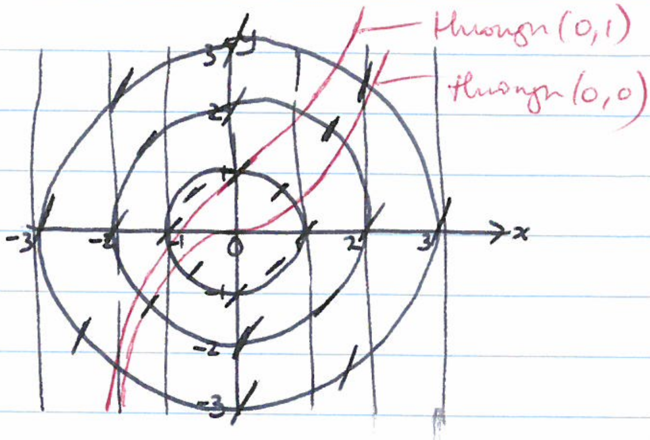
$\sqrt{x^2+y^2} = 1 \Rightarrow x^2+y^2 = 1$ a circle centre $(0,0)$ radius 1.

(ii) isoclines for $\frac{dy}{dx} = 1$ circle rad 1

" $\frac{dy}{dx} = 2$ circle rad 2

" $\frac{dy}{dx} = 3$ circle rad 3

So circle rad 1 – all coors have grad 1 etc.



(iii) Solution curve through $(0,1)$

grad at $(0,1)$ is 1

grad at $(1,2)$ is $\sqrt{5}$

(iv) solution curve through $(0,0)$

grad at $(0,0)$ is 0 – point of inflexion.

3(a) $O \longrightarrow \longleftarrow 2k^2x$, $\leftarrow x \rightarrow$

Using 'F=ma' in direction of motion

$-2k^2x = ma$

but $a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot v$

$\therefore mv\frac{dv}{dx} = -2k^2x$

$\therefore 2v\frac{dv}{dx} = -2k^2x \Rightarrow v\frac{dv}{dx} = -k^2x$

(ii) $\therefore \int v\,dv = -k^2\int x\,dx$

$\therefore \frac{v^2}{2} = -k^2\frac{x^2}{2} + c \qquad \therefore v^2 = -k^2x^2 + A$

$0 = -k^2a^2 + A$

when $x = a$ $v = 0$ $\qquad \therefore A = k^2a^2$

$\therefore v^2 = -k^2x^2 + k^2a^2$

$\therefore v^2 = k^2(a^2 - x^2)$

$\therefore v = \pm k\sqrt{a^2-x^2}$

If the particle is moving in the negative direction $v<0$

so $v = -k\sqrt{a^2-x^2}$ and since $v = \frac{dx}{dt}$

then $\frac{dx}{dt} = -k\sqrt{a^2-x^2}$

(iii) when $t=0$ $x=a$. If $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+c_1$

then $\int \frac{dx}{\sqrt{a^2-x^2}} = -k\int dt$ gives $\arcsin\left(\frac{x}{a}\right) = -kt + c$

when $t=0$ $x=a$ so $c = \arcsin(1) = \frac{\pi}{2}$ since $\sin\left(\frac{\pi}{2}\right)=1$

$\therefore \frac{x}{a} = \sin\left(\frac{\pi}{2}-kt\right)$

$\therefore x = a\sin\left(\frac{\pi}{2}-kt\right)$

(b) (i) $\omega\frac{d\omega}{d\theta} = -9\sin\theta$ where $\omega = \frac{d\theta}{dt}$ and $\omega=0$ when $\theta = \frac{\pi}{3}$

$\therefore \int \omega\, d\omega = -9\int \sin\theta\, d\theta$

$\therefore \frac{\omega^2}{2} = 9\cos\theta + k$

$\omega = 0$ when $\theta = \frac{\pi}{3}$ $\therefore$ $9\cos\left(\frac{\pi}{3}\right)+k=0$ $\Rightarrow$ $k = -\frac{9}{2}$

$\therefore \frac{\omega^2}{2} = 9\cos\theta - \frac{9}{2}$ $\Rightarrow$ $\omega^2 = 18\cos\theta - 9$ $= 9(2\cos\theta - 1)$

$\therefore$ $\omega = \pm\sqrt{9(2\cos\theta-1)}$ $=\pm 3(2\cos\theta-1)^{1/2}$

and $\omega = \frac{d\theta}{dt}$ $\therefore$ $\frac{d\theta}{dt} = \pm 3\sqrt{2\cos\theta - 1}$

For θ decreasing $\frac{d\theta}{dt} = -3\sqrt{2\cos\theta-1}$

(ii) Euler's Method $\theta = \frac{\pi}{3}$ when $t=0.0$ $h=0.1$ estimate $\theta(0.1)$

t	θ	$\frac{d\theta}{dt}$	$h\frac{d\theta}{dt}$	$\theta + h\frac{d\theta}{dt}$
0.0	$\frac{\pi}{3}$	0	0	$\pi/3$
0.1	$\frac{\pi}{3}$			

All subsequent iterations will continue to give $\pi/3$ as the substitution for $\theta = \pi/3$ in $\frac{d\theta}{dt}$ will give 0.

4 (i) $\frac{dx}{dt} = -3x - 2y + 3t$ $\quad \therefore\ 2y = -\dot{x} - 3x + 3t$

$\therefore\ 2\dot{y} = -\ddot{x} - 3\dot{x} + 3$

$\frac{dy}{dt} = 2x + y + t + 2$ $\quad y = \dot{y} - 2x - t - 2$

$\therefore\ \dot{y} = \frac{1}{2}(-\dot{x} - 3x + 3t) + 2x + t + 2 = \frac{1}{2}(-\ddot{x} - 3\dot{x} + 3)$

$\therefore\ -\dot{x} - 3x + 3t + 4x + 2t + 4 = -\ddot{x} - 3\dot{x} + 3$

$\therefore\ \ddot{x} + 2\dot{x} + x = -5t - 1$

$\therefore\ \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = -5t - 1$

(ii) Auxilliary Equation from LHS

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda + 1)^2 = 0$

$\lambda = -1$ Repeated

$\therefore$ CF will be $x = Ae^{-t} + Bte^{-t}$

Particular Integral from RHS

Since RHS $= -5t - 1$

assume PI $x = Ct + D$

$\therefore\ \dot{x} = C \quad \ddot{x} = 0$

$\therefore\ 2C + Ct + D = -5t - 1$

$\therefore\ C = -5$ and $D = 9$

GS = CF + PI

$\therefore$ General Solution is $x = Ae^{-t} + Bte^{-t} - 5t + 9$

(iii) For y then : $y = \frac{1}{2}(-\dot{x} - 3x + 3t)$ but $\dot{x} = -Ae^{-t} - Bte^{-t} + Be^{-t} - 5$

$\therefore\ y = -\frac{1}{2}(-Ae^{-t} - Bte^{-t} + Be^{-t} - 5) - \frac{3}{2}x + \frac{3}{2}t$

$\therefore\ y = \frac{A}{2}e^{-t} + \frac{B}{2}te^{-t} - \frac{B}{2}e^{-t} + \frac{5}{2} - \frac{3}{2}x + \frac{3}{2}t$

$\therefore\ y = \frac{1}{2}(A + Bt)e^{-t} - \frac{B}{2}e^{-t} + \frac{5}{2} + \frac{3}{2}t - \frac{3}{2}(Ae^{-t} + Bte^{-t} - 5t + 9)$ ✓ ✓ ✓ ✓ ✓ ✓

$\therefore\ y = -Ae^{-t} - Bte^{-t} - \frac{B}{2}e^{-t} - \frac{22}{2} + 9t$

$\therefore\ y = -e^{-t}(A + Bt + \frac{B}{2}) - 11 + 9t$

(iv) When $t = 0$, $x = 9$ and $y = 0$

$\therefore\ x = 9 = A + 9 \Rightarrow A = 0$

$y = 0 = -\frac{B}{2} - 11 \Rightarrow B = -22$

Particular Solutions

$x = -22te^{-t} - 5t + 9$

$y = -e^{-t}(-22t - 11) - 11 + 9t$

$\therefore\ y = 11e^{-t}(2t + 1) - 11 + 9t$

(v) When $t \to \infty$ $e^{-t} \to 0$ $\therefore\ x \to 9 - 5t$

and $y \to 9t - 11$