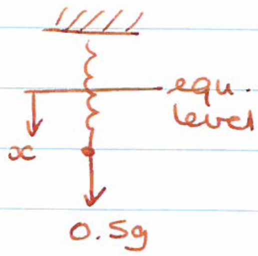
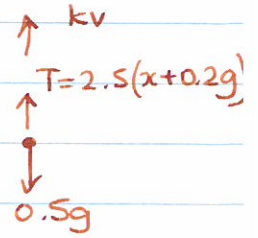
DE January 2012.





1.(i) using "F = ma" for downwards motion

$mg - T - R = ma$

$\therefore\ 0.5g - 2.5(x+0.2g) - kv = 0.5a$

but $v = \frac{dx}{dt}$ $\quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

$\therefore\ 0.5g - 2.5(x+0.2g) - k\frac{dx}{dt} = 0.5\frac{d^2x}{dt^2}$

$\therefore\ 0.5\frac{d^2x}{dt^2} + k\frac{dx}{dt} + 2.5x = 0$

$\therefore\ \frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 5x = 0$

(ii) $t=0$ $\quad x = 0.1$

For damping - overdamped $b^2 - 4ac > 0$ (no oscillating)

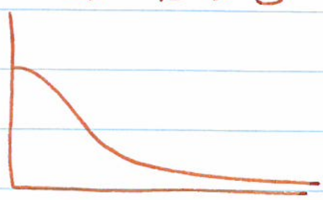
critical damping $b^2 - 4ac = 0$ (osc. ~~continuously~~)

under damped $b^2 - 4ac < 0$ (osc ~~stop fast~~)

(A) for overdamping

$(2k)^2 - 4(1)(5) > 0 \Rightarrow 4k^2 > 20 \Rightarrow k^2 > 5$

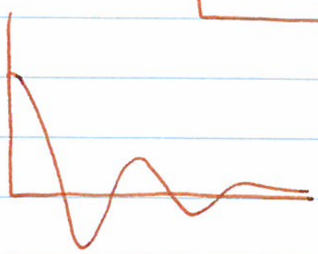
Since $k > 0$ then $k > \sqrt{5}$



(B) for underdamping

$0 < k^2 < 5$ since $k > 0$

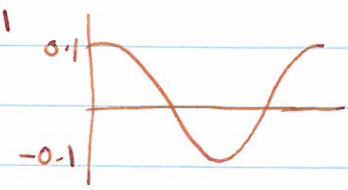
$\therefore\ 0 < k < \sqrt{5}$



(C) for critical damping

$k = \sqrt{5}$ and oscillates continuously

(iii) If $k=0$ then no damping and since $t=0$ $x=0.1$ the particle oscillates between 0.1 and -0.1 continuously



(iv) $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = \sin 4t$

if $t=0$ and $x=0$ initial conditions are particle is at rest $(v=0)$

Auxiliary equation

$\lambda^2 + 2\lambda + 5 = 0$

$(\lambda+1)^2 - 1 + 5 = 0$

$(\lambda+1)^2 = -4$

$(\lambda+1)^2 = 4j^2$

$\lambda+1 = \pm 2j$

$\lambda = -1 \pm 2j$

$\therefore$ Complementary Function is

$x = e^{-t}(A\cos 2t + B\sin 2t)$

$\therefore GS = CF + PI$

$\therefore x = e^{-t}(A\cos 2t + B\sin 2t)$

$- \frac{8}{185}\cos 4t - \frac{11}{185}\sin 4t$

Use the condition $t=0$, $x=0$

$\therefore 0 = A - \frac{8}{185} \Rightarrow A = \frac{8}{185}$

Particular integral based on RHS being $\sin 4t$

$\therefore x = C\cos 4t + D\sin 4t$

$\therefore \dot{x} = -4C\sin 4t + 4D\cos 4t$

$\therefore \ddot{x} = -16C\cos 4t - 16D\sin 4t$

Substitute into original DE

$-16C\cos 4t - 16D\sin 4t$

$- 8C\sin 4t + 8D\cos 4t$

$+ 5C\cos 4t + 5D\sin 4t = \sin 4t$

cf: $\sin 4t$

$(-16D) + (-8C) + (5D) = 1$

cf: $\cos 4t$

$(-16C) + (+8D) + (5C) = 0$

$\therefore -8C - 11D = 1$ $\quad \times 8$

$-11C + 8D = 0$ $\quad \times 11$

add

$-185C = 8$ $\quad \therefore C = \frac{-8}{185}$

$\therefore D = -\frac{11}{185}$

Since particle at rest $v = \dot{x} = 0$

$\dot{x} = -e^{-t}(A\cos 2t + B\sin 2t) + e^{-t}(-2A\sin 2t + 2B\cos 2t)$

$- \frac{32}{185}\sin 4t + \frac{44}{185}\cos 4t$

So $t=0$ $\dot{x}=0$

$\therefore 0 = -A + 2B + \frac{44}{185} \Rightarrow B = \frac{26}{185}$

Final solution is

$x = e^{-t}\left(\frac{8}{185}\cos 2t + \frac{26}{185}\sin 2t\right) - \frac{8}{185}\cos 4t - \frac{11}{185}\sin 4t$

2.(i) $t=0 \quad P=1000$
$t=10 \quad P=4000$ $\Big\}$ let $\frac{dP}{dt}$ be rate of growth of population

$\therefore \frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$

$\therefore \int \frac{dP}{P} = \int k\,dt \Rightarrow \ln P = kt + c$

When $t=0$ $P=1000$ $\quad \ln 1000 = c$

when $t=10$ $P=4000$ $\quad \ln 4000 = 10k + \ln 1000$

$\therefore \ln 4 = 10k$

$\therefore k = \frac{1}{10}\ln 4$

$\therefore \ln P = \frac{1}{10} t \ln 4 + \ln 1000 \qquad e^{\ln 4^{t/10}} \cdot e^{\ln 1000}$

$\therefore P = 4^{0.1t} \times 1000$

(ii) $\frac{dP}{dt} = kP(5000 - P)$ (solve for t) $\Rightarrow \int \frac{dP}{P(5000-P)} = \int k\,dt$

Consider partial fractions

$\frac{1}{P(5000-P)} \equiv \frac{A}{P} + \frac{B}{(5000-P)}$

$\therefore A(5000-P) + BP \equiv 1$

when $P = 5000$ $\quad B = \frac{1}{5000}$
When $P = 0$ $\quad A = \frac{1}{5000}$ $\Big\}$ $\therefore \frac{1}{P(5000-P)} \equiv \frac{1}{5000P} + \frac{1}{5000(5000-P)}$

$\therefore \frac{1}{5000} \int \left\{ \frac{1}{P} + \frac{1}{(5000-P)} \right\} dP = kt + c$

$\therefore \ln P + (-1) \ln(5000-P) = 5000(kt+c)$

$\therefore \ln\left(\frac{P}{5000-P}\right) = 5000(kt+c)$

when $t=0$ $P=1000$ $\quad \ln\left(\frac{1}{4}\right) = 5000c \Rightarrow c = \frac{1}{5000}\ln(1/4)$

When $t=10$ $P=4000$ $\quad \ln(4) = 5000\left\{10k + \frac{1}{5000}\ln(1/4)\right\}$

$\therefore \ln 4 - \ln\left(\frac{1}{4}\right) = 50000k$

$\therefore k = \frac{1}{50000}\ln 16$

$\therefore \ln\left(\frac{P}{5000-P}\right) = \left(\frac{1}{10}\ln 16\right)t + \ln\left(\frac{1}{4}\right)$

$\therefore t = \left(\frac{10}{\ln 16}\right)\ln\left(\frac{4P}{5000-P}\right)$

(iii) If $t = \left(\frac{10}{\ln 16}\right) \ln\left(\frac{4P}{5000-P}\right)$

If $P = 4900$ $\quad t = \left(\frac{10}{\ln 16}\right) \ln\left(\frac{4 \times 49\cancel{00}}{1\cancel{00}}\right) = 19.036774 \quad = 19.04$ (2dp)

(iv) $\frac{dP}{dt} = 10^{-15} P^{\alpha}(5000 - P)$ $\quad$ max rate of growth when $P = 4000$

$\frac{dP}{dt}$ has stationary point when $\frac{d^2P}{dt^2} = 0$

$$\frac{d}{dt}\left(10^{-15} P^{\alpha}(5000 - P)\right) = 10^{-15} \alpha P^{\alpha-1} \times 5000 - 10^{-15}(\alpha+1) P^{\alpha}$$

$$= 10^{-15} P^{\alpha-1}(5000\alpha - P\alpha - P)$$

If $10^{-15} P^{\alpha-1}(5000\alpha - P\alpha - P) = 0$ at st pt

[Then for max growth rate

$P^{\alpha-1}(5000\alpha - P\alpha - P) < 0$]

Since $P > 0$ then $5000\alpha - P\alpha - P = 0$ when $P = 4000$

$$5000\alpha = (\alpha+1)4000$$

$$1000\alpha = 4000$$

$$\therefore \alpha = 4.$$

(v) Euler's method $h = 0.2$ Find $P(10.4)$ starting from $t = 10, P = 4000$

t	P	$\frac{dP}{dt}$	$h\frac{dP}{dt}$	$P + h\frac{dP}{dt}$
10	4000	256	51.2	4051.2
10.2	4051.2	255.5697288	51.11394575	4102.313946
10.4	4102.314			

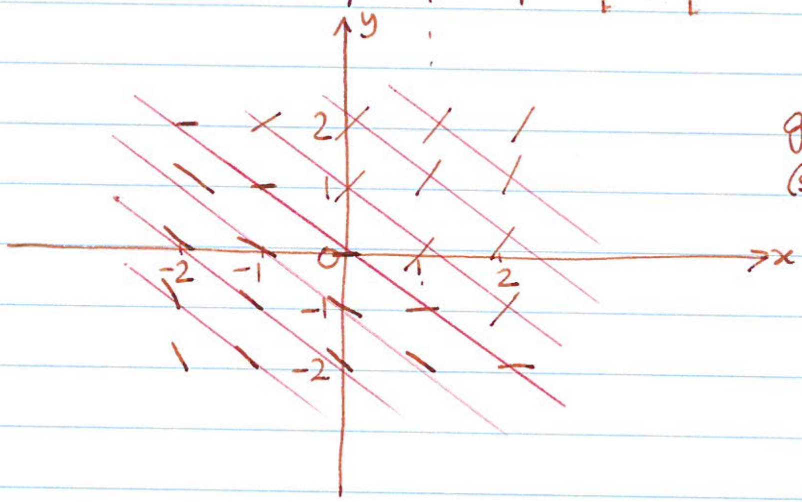
So $P \approx 4102$ when $t = 10.4$

3 (i) $\frac{dy}{dx} - y = x$

$\therefore \frac{dy}{dx} = x + y$

$y \backslash x$	−2	−1	0	1	2
−2	−4	−3	−2	−1	0
−1	−3	−2	−1	0	1
0	−2	−1	0	1	2
1	−1	0	1	2	3
2	0	1	2	3	4

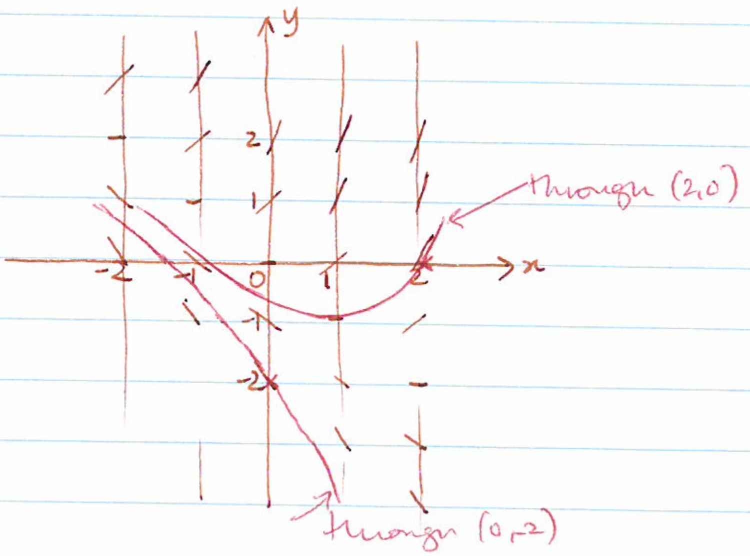
isoclines (gradient function)



graph of isoclines (same gradients joined up in this case by straight lines)

(ii) Graph cannot cross the isoclines - the direction indicators show that the curves have min tp above the line where $\frac{dy}{dx} = -1$ so curves have asymptote $x+y=-1$

(iii)



(iv) $\dfrac{dy}{dx} - y = x$ when $x = 0,\ y = 3$ cf $\dfrac{dy}{dx} + P(x)y = Q(x)$

Integrating Factor $= e^{\int P(x)\,dx} = e^{-\int dx} = e^{-x}$

$\therefore \dfrac{d}{dx}\left(e^{-x}y\right) = e^{-x}x$ (multiplying through by e^{-x})

$\therefore e^{-x}y = \int e^{-x}x\,dx$

$u = x \quad du = 1$

$dv = e^{-x} \quad v = -e^{-x}$

$\int u\,dv = uv - \int v\,du$

$\therefore e^{-x}y = -xe^{-x} + \int e^{-x}\,dx$

$\therefore e^{-x}y = -xe^{-x} - e^{-x} + c$

when $x = 0,\ y = 3$

$\therefore 3 = -1 + c \Rightarrow c = 4$

$\therefore e^{-x}y = -xe^{-x} - e^{-x} + 4$

$\therefore y = -x - 1 + 4e^{x}$

(v) $\dfrac{dy}{dx} - y = \sin x$

Auxiliary Eq. LHS let $y = Ae^{\lambda x}$

$\therefore \lambda - 1 = 0$

$\therefore \lambda = 1$

$\therefore$ CF $y = Ae^{x}$

Particular Integral RHS

let $y = C\cos x + D\sin x$

$\therefore y' = -C\sin x + D\cos x$

Substituting into DE then

$-C\sin x + D\cos x - C\cos x - D\sin x = \sin x$

cf $\sin x$: $-C - D = 1$

cf $\cos x$: $D - C = 0$

$\therefore C = D = -\frac{1}{2}$

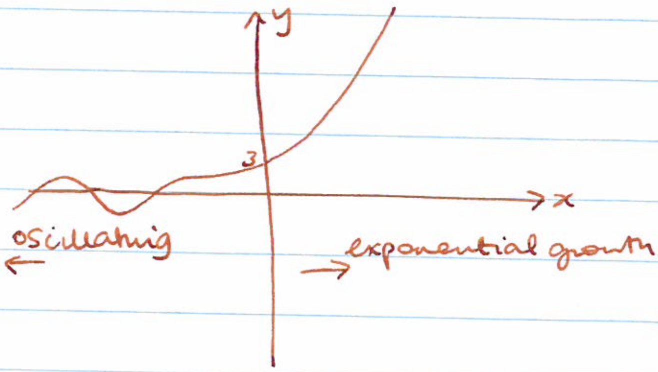
$\therefore$ General Solution $=$ CF $+$ PI

$\therefore y = Ae^{x} - \frac{1}{2}\cos x - \frac{1}{2}\sin x$

(vi) If $y = 3$ when $x = 0$

$3 = A - \frac{1}{2} \Rightarrow A = \frac{7}{2}$

$\therefore y = \frac{7}{2}e^{x} - \frac{1}{2}\cos x - \frac{1}{2}\sin x$



4.(i) $\frac{dx}{dt} = -x+2y \Rightarrow \dot{x} = -x+2y \Rightarrow y = \frac{1}{2}\dot{x} + \frac{1}{2}x$

$\frac{dy}{dt} = -x-4y+e^{-2t} \quad \dot{y} = -x-4y+e^{-2t} \qquad \therefore \dot{y} = \frac{1}{2}\ddot{x} + \frac{1}{2}\dot{x}$

$\therefore \frac{1}{2}\ddot{x} + \frac{1}{2}\dot{x} = -x-4y+e^{-2t}$

$\therefore \frac{1}{2}\ddot{x} + \frac{1}{2}\dot{x} + x + 4\left(\frac{1}{2}\dot{x} + \frac{1}{2}x\right) = e^{-2t}$

$\therefore \ddot{x} + \dot{x} + 2x + 4\dot{x} + 4x = 2e^{-2t}$

$\therefore \ddot{x} + 5\dot{x} + 6x = 2e^{-2t}$

Auxiliary Equation from LHS

$\lambda^2 + 5\lambda + 6 = 0$

$(\lambda+3)(\lambda+2) = 0$

$\therefore \lambda = -3$ or $\lambda = -2$

$\therefore$ Complementary Function

$x = Ae^{-3t} + Be^{-2t}$

Particular integral from RHS

Since Be^{-2t} already forms part of solution, choose

$x = Cte^{-2t}$

$\therefore \dot{x} = Ce^{-2t} - 2Cte^{-2t}$

$\ddot{x} = -2Ce^{-2t} - 2Ce^{-2t} + 4Cte^{-2t}$

Substitute into DE

$-4Ce^{-2t} + 4Cte^{-2t} + 5Ce^{-2t} - 10Cte^{-2t} + 6Cte^{-2t} = 2e^{-2t}$

cf e^{-2t} $\quad -4C + 5C = 2 \Rightarrow C = 2$

General Solution = CF + PI

$\therefore x = Ae^{-3t} + Be^{-2t} + 2te^{-2t}$

(ii) $\therefore \dot{x} = -3Ae^{-3t} - 2Be^{-2t} + 2e^{-2t} - 4te^{-2t}$

$y = \frac{1}{2}\dot{x} + \frac{1}{2}x$

$\therefore y = -\frac{3}{2}Ae^{-3t} - Be^{-2t} + e^{-2t} - 2te^{-2t} + \frac{A}{2}e^{-3t} + \frac{B}{2}e^{-2t} + te^{-2t}$

$\therefore y = -Ae^{-3t} - \frac{B}{2}e^{-2t} + e^{-2t} - te^{-2t}$ is the general solution for y.

(iii) If $x=5$ and $y=0$ initially (when $t=0$)

$\left.\begin{array}{l} \therefore 5 = A + B \quad \text{(for } x\text{)} \\ \& \ 0 = -A - \frac{B}{2} + 1 \quad \text{(for } y\text{)} \Rightarrow A = 1 - \frac{B}{2} \end{array}\right\} \therefore B = 8, \ A = -3$

$\therefore x = -3e^{-3t} + 8e^{-2t} + 2te^{-2t}$

$\& \ y = 3e^{-3t} - 4e^{-2t} + e^{-2t}(1-t)$

$\therefore y = 3e^{-3t} - 3e^{-2t} - te^{-2t}$

(iv) As $t \to \infty$ $\quad e^{-nt} \to 0$

$$\frac{y}{x} = \frac{3e^{-3t} - 3e^{-2t} - te^{-2t}}{-3e^{-3t} + 8e^{-2t} + 2te^{-2t}} = \frac{3e^{-t} - 3 - t}{-3e^{-t} + 8 + 2t}$$

As $t \to \infty$

$$\frac{y}{x} \to \frac{-t}{2t} \quad \text{so} \quad \frac{y}{x} \to -\frac{1}{2}$$

If $\frac{y}{x} = -\frac{1}{2}$ then $2y = -x$

$$\therefore 2(3e^{-t} - 3 - t) = -(-3e^{-t} + 8 + 2t)$$

$$\therefore 6e^{-t} - 6 - 2t = 3e^{-t} - 8 - 2t$$

$$\therefore 3e^{-t} = -2$$

$$\therefore e^{-t} = -\frac{2}{3}$$

However $e^{-t} > 0$ for all t.

So there is no solution for t for which $\frac{y}{x} = -\frac{1}{2}$.