FP2 January 2006.

3 (i) $M = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 6 \\ 2 & 2 & -4 \end{pmatrix}$ for the characteristic equation we need to calculate $\det(M - \lambda I) = 0$

$\det(M-\lambda I)$ from $(M - \lambda I) = \begin{pmatrix} 1-\lambda & 2 & 3 \\ -2 & -3-\lambda & 6 \\ 2 & 2 & -4-\lambda \end{pmatrix}$

$\det(M-\lambda I)$

$= (1-\lambda)\left[(3+\lambda)(4+\lambda) - 12\right] - 2\left[2(4+\lambda) - 12\right] + 3\left[-4 + 2(3+\lambda)\right]$

$= (1-\lambda)(\lambda^2 + 7\lambda) - 2(2\lambda - 4) + 3(2\lambda + 2)$

$= -\lambda^3 - 7\lambda^2 + \lambda^2 + 7\lambda + 8 - 4\lambda + 6 + 6\lambda$

$= -\lambda^3 - 6\lambda^2 + 9\lambda + 14$

for $\det(M - \lambda I) = 0$ then $-\lambda^3 - 6\lambda^2 + 9\lambda + 14 = 0$

$\therefore \lambda^3 + 6\lambda^2 - 9\lambda - 14 = 0.$

(ii) If -1 is an eigenvalue of M, then $\lambda = -1$ should give CE $= 0$

$\therefore (-1)^3 + 6(-1)^2 - 9(-1) - 14 = -1 + 6 + 9 - 14 = 0$

$\therefore -1$ is an eigenvalue of $M \Rightarrow (\lambda + 1)$ is a factor of CE

$\therefore (\lambda+1)(\lambda^2 + 5\lambda - 14) = 0$

$(\lambda+1)(\lambda+7)(\lambda-2) = 0$

$\therefore \lambda = -1, \ \lambda = -7, \ \lambda = 2$

The other eigenvalues are -7 and 2.

(iii) Eigenvector when $\lambda = -1$ $(M - \lambda I)s = 0$

$\begin{pmatrix} 2 & 2 & 3 \\ -2 & -2 & 6 \\ 2 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$2x + 2y + 3z = 0$ ①

$-2x - 2y + 6z = 0$ ②

$2x + 2y - 3z = 0$ ③

① - ③ gives $6z = 0$ $\therefore z = 0$

$\therefore 2x + 2y = 0$ so let $x = 1$ then $y = -1$

so an eigenvector corresponding to $\lambda = -1$ would be $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(iv). If $\underline{s}$ is an eigenvector then $MS = \lambda I S$ from before $(M-\lambda I)s=0$

$$\therefore MS_1 = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 6 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3+3 \\ -6+6 \\ 6-4 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

Also

$$MS_2 = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 6 \\ 2 & 2 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 6-6 \\ -9-12 \\ 6+8 \end{pmatrix} = \begin{pmatrix} 0 \\ -21 \\ 14 \end{pmatrix} = -7\begin{pmatrix} 0 \\ +3 \\ -2 \end{pmatrix}$$

So both S_1 and S_2 are eigenvectors of M. (eigenvalues 2 and -7)

(v) P and diagonal matrix D such that $M^3 = PDP^{-1}$

For $M^n = S\Lambda^n S^{-1}$ S is matrix of eigenvectors corresponding to Λ matrix of eigenvalues.

$$\therefore \text{let } P = \begin{pmatrix} 1 & 3 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{where } D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -7 \end{pmatrix}^3$$

$$\therefore D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -343 \end{pmatrix}$$ (notice $D = \Lambda^3$)

(vi) Cayley Hamilton Theorem – M satisfies its own Characteristic Eq

CE : $\lambda^3 + 6\lambda^2 - 9\lambda - 14 = 0$

$\therefore$ $M^3 + 6M^2 - 9M - 14I = 0$

$\therefore M^2 + 6M - 9I - 14M^{-1} = 0$

$\therefore M^2 + 6M - 9I = 14M^{-1}$

$\therefore M^{-1} = \frac{1}{14}M^2 + \frac{6}{14}M - \frac{9}{14}I$

FP2. June 2006.

3. (i) Inverse of $\begin{pmatrix} 4 & 1 & k \\ 3 & 2 & 5 \\ 8 & 5 & 13 \end{pmatrix}$ where $k \neq 5$

$$M^{-1} = \frac{1}{\det M}(\operatorname{adj} M) = \frac{1}{\det M}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \quad \text{for } \det M \neq 0$$

$$+A_1 = \begin{vmatrix} 2 & 5 \\ 5 & 13 \end{vmatrix} = 26 - 25 = 1$$

$$-A_2 = \begin{vmatrix} 1 & k \\ 5 & 13 \end{vmatrix} = 13 - 5k$$

$$+A_3 = \begin{vmatrix} 1 & k \\ 2 & 5 \end{vmatrix} = 5 - 2k$$

$$\begin{aligned} \therefore \det M &= (4)A_1 - (3)A_2 + (8)A_3 \\ &= 4 - 3(13 - 5k) + 8(5 - 2k) \\ &= 4 - 39 + 15k + 40 - 16k \\ &= 5 - k \end{aligned}$$

$$-B_1 = \begin{vmatrix} 3 & 5 \\ 8 & 13 \end{vmatrix} = 39 - 40 = -1 \qquad +C_1 = \begin{vmatrix} 3 & 2 \\ 8 & 5 \end{vmatrix} = 15 - 16 = -1$$

$$+B_2 = \begin{vmatrix} 4 & k \\ 8 & 13 \end{vmatrix} = 52 - 8k \qquad -C_2 = \begin{vmatrix} 4 & 1 \\ 8 & 5 \end{vmatrix} = 20 - 8 = 12$$

$$-B_3 = \begin{vmatrix} 4 & k \\ 3 & 5 \end{vmatrix} = 20 - 3k \qquad +C_3 = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 8 - 3 = 5$$

$$\therefore M^{-1} = \frac{1}{(5-k)}\begin{pmatrix} 1 & -(13-5k) & +(5-2k) \\ -(-1) & +(52-8k) & -(20-3k) \\ +(-1) & -(12) & +5 \end{pmatrix}$$

$$\therefore M^{-1} = \frac{1}{(5-k)}\begin{pmatrix} 1 & 5k-13 & 5-2k \\ 1 & 52-8k & 3k-20 \\ -1 & -12 & 5 \end{pmatrix}$$

(ii)
$$\begin{aligned} 4x + y + 7z &= 12 \\ 3x + 2y + 5z &= m \\ 8x + 5y + 13z &= 0 \end{aligned} \Rightarrow \begin{pmatrix} 4 & 1 & 7 \\ 3 & 2 & 5 \\ 8 & 5 & 13 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ m \\ 0 \end{pmatrix} \quad \text{so } k = 7.$$

If $k=7$ then $M^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & 22 & -9 \\ 1 & -4 & 1 \\ -1 & -12 & 5 \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 22 & -9 \\ 1 & -4 & 1 \\ -1 & -12 & 5 \end{pmatrix}\begin{pmatrix} 12 \\ m \\ 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 12+22m \\ 12-4m \\ -12-12m \end{pmatrix} = \begin{pmatrix} -6-11m \\ -6+2m \\ 6+6m \end{pmatrix}$

$\therefore \quad x = -6-11m \quad ; \quad y = -6+2m \quad ; \quad z = 6+6m$

(iii) ① $4x + y + 5z = 12$
② $3x + 2y + 5z = p$
③ $8x + 5y + 13z = 0$

$\Rightarrow \begin{pmatrix} 4 & 1 & 5 \\ 3 & 2 & 5 \\ 8 & 5 & 13 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ p \\ 0 \end{pmatrix} \Rightarrow k=5$

Which means that the determinant is now 0; (cannot use M^{-1})

So do ③ − 2①
$8x + 2y + 10z = 24$
$8x + 5y + 13z = 0$
$\Big\}\quad 3y + 3z = -24$
$y + z = -8 \quad$ ④

and do 3① − 4②
$12x + 3y + 15z = 36$
$12x + 8y + 20z = 4p$
$\Big\}\quad 5y + 5z = 4p - 36 \quad$ ⑤

$\therefore \quad 4p - 36 = 5y + 5z = 5(y+z) = 5(-8)$

$\therefore \quad 4p = 36 - 40$

$\therefore p = -1$

For general solution, using a parameter for one variable

let $z = \lambda$, then from ④ $z = -y-8 \Rightarrow y = -8-\lambda$

using $4x = 12 - y - 5z$

$= 12 - (-8-\lambda) - 5\lambda$

$= 20 - 4\lambda$

$\therefore x = 5 - \lambda$

So general solution will be $x = 5-\lambda$; $y = -8-\lambda$; $z = \lambda$.

FP2 January 2007.

3. $P = \begin{pmatrix} 4 & 2 & k \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ $k \neq 4$ and $M = \begin{pmatrix} 2 & -2 & -6 \\ -1 & 3 & 1 \\ 1 & -2 & -2 \end{pmatrix}$

(i) P^{-1} : $\det P = 4\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$

$= 4(-1) - 2(-4) + k(-1)$

$= 4 - k$

$A_1 = \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix}$ $A_2 = -\begin{vmatrix} 2 & k \\ 0 & -1 \end{vmatrix}$ $A_3 = \begin{vmatrix} 2 & k \\ 1 & 3 \end{vmatrix}$

$B_1 = -\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}$ $B_2 = \begin{vmatrix} 4 & k \\ 1 & -1 \end{vmatrix}$ $B_3 = -\begin{vmatrix} 4 & k \\ 1 & 3 \end{vmatrix}$

$C_1 = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$ $C_2 = -\begin{vmatrix} 4 & 2 \\ 1 & 0 \end{vmatrix}$ $C_3 = \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}$

$\therefore P^{-1} = \dfrac{1}{\det P}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

$\therefore P^{-1} = \dfrac{1}{(4-k)}\begin{pmatrix} -1 & 2 & 6-k \\ 4 & -4-k & -12+k \\ -1 & 2 & 2 \end{pmatrix}$

And when $k = 2$ $P^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$

(ii) For an eigenvector s $Ms = \lambda I s$ [or $(M - \lambda I)s = 0$]

$\therefore \begin{pmatrix} 2 & -2 & -6 \\ -1 & 3 & 1 \\ 1 & -2 & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ $\therefore$ eigenvalue $= 0$

$\begin{pmatrix} 2 & -2 & -6 \\ -1 & 3 & 1 \\ 1 & -2 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\therefore$ eigenvalue $= 1$

$\begin{pmatrix} 2 & -2 & -6 \\ -1 & 3 & 1 \\ 1 & -2 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ $\therefore$ eigenvalue $= 2$.

(iii) $M^n = S \Lambda^n S^{-1}$ where S is matrix of eigenvectors and Λ is matrix of eigenvalues.

$$\therefore M^n = \begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^n \frac{1}{2} \begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$$

[$S = P$ in (i) where $k = 2$.]

$$\therefore M^n = \frac{1}{2} \begin{pmatrix} 0 & 2 & 2^{n+1} \\ 0 & 1 & 3 \times 2^n \\ 0 & 0 & -2^n \end{pmatrix} \begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$$

$$\therefore M^n = \frac{1}{2} \begin{pmatrix} 8 - 2^{n+1} & -12 + 2^{n+2} & -20 + 2^{n+2} \\ 4 - 3 \times 2^n & -6 + 3 \times 2^{n+1} & -10 + 3 \times 2^{n+1} \\ 2^n & -2^{n+1} & -2^{n+1} \end{pmatrix}$$

$$\therefore M^n = \begin{pmatrix} 4 - 2^n & -6 + 2^{n+1} & -10 + 2^{n+1} \\ 2 - 3 \times 2^{n-1} & -3 + 3 \times 2^n & -5 + 3 \times 2^n \\ 2^{n-1} & -2^n & -2^n \end{pmatrix}$$

$$\therefore M^n = \begin{pmatrix} 4 & -6 & -10 \\ 2 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix} + 2^{n-1} \begin{pmatrix} -2 & 4 & 4 \\ -3 & 6 & 6 \\ 1 & -2 & -2 \end{pmatrix}$$

FP2 June 2007.

3. (i) $M = \begin{pmatrix} 3 & 5 & 2 \\ 5 & 3 & -2 \\ 2 & -2 & -4 \end{pmatrix}$ $\quad (M - \lambda I) = \begin{pmatrix} 3-\lambda & 5 & 2 \\ 5 & 3-\lambda & -2 \\ 2 & -2 & -4-\lambda \end{pmatrix}$

characteristic equation is given by $\det(M - \lambda I) = 0$

$\therefore (3-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ -2 & -4-\lambda \end{vmatrix} - 5\begin{vmatrix} 5 & -2 \\ 2 & -4-\lambda \end{vmatrix} + 2\begin{vmatrix} 5 & 3-\lambda \\ 2 & -2 \end{vmatrix} = 0$

$\therefore (3-\lambda)\left[(3-\lambda)(-4-\lambda) - 4\right] - 5\left[-20 - 5\lambda + 4\right] + 2\left[-10 - 6 + 2\lambda\right] = 0$

$(3-\lambda)(\lambda^2 + \lambda - 16) - 5(-5\lambda - 16) + 2(2\lambda - 16) = 0$

$\therefore 3\lambda^2 + 3\lambda - 48 - \lambda^3 - \lambda^2 + 16\lambda + 25\lambda + 80 + 4\lambda - 32 = 0$

$\therefore -\lambda^3 + 2\lambda^2 + 48\lambda = 0$

$\therefore \lambda^3 - 2\lambda^2 - 48\lambda = 0$

(ii) $\lambda(\lambda^2 - 2\lambda - 48) = 0$

$\therefore \lambda(\lambda + 6)(\lambda - 8) = 0$

$\therefore \lambda = 0$ or $\lambda = -6$ or $\lambda = 8$

$\therefore$ eigenvalues are $0, -6$ and 8

when $\lambda = -6$ $\quad (M - \lambda I) = \begin{pmatrix} 9 & 5 & 2 \\ 5 & 9 & -2 \\ 2 & -2 & 2 \end{pmatrix}$

$\therefore \begin{pmatrix} 9 & 5 & 2 \\ 5 & 9 & -2 \\ 2 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$9x + 5y + 2z = 0$ ① $\qquad$ ① − ③ $\Rightarrow 7x + 7y = 0 \qquad x = -y$

$5x + 9y - 2z = 0$ ② $\qquad$ 2② − 5③ $\Rightarrow 28y - 14z = 0 \qquad z = 2y$

$2x - 2y + 2z = 0$ ③

$\therefore$ eigenvector corresponding to eigenvalue -6 is $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

when $\lambda = 8$ $\quad \begin{pmatrix} -5 & 5 & 2 \\ 5 & -5 & -2 \\ 2 & -2 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-5x + 5y + 2z = 0$ ① $\qquad$ 2② − 5③ $\Rightarrow z = 0$

$5x - 5y - 2z = 0$ ② $\qquad \therefore 5y = 5x \qquad x = y$

$2x - 2y - 12z = 0$ ③ $\qquad$ eigenvector for 8 is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(iii) If $P^{-1}M^2P = D$ using $M^n = S\Lambda^n S^{-1}$ S eigenvector matrix
Λ eigenvalue matrix

$$P = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 8 \end{pmatrix}^2$$

since $n=2$

$$\therefore D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 64 \end{pmatrix}$$

(iv) Cayley Hamilton Theorem – M satisfies its own char. equation.

$$\lambda^3 - 2\lambda^2 - 48\lambda = 0$$

$$\therefore M^3 - 2M^2 - 48M = 0 \qquad M^3 = 2M^2 + 48M$$

$$\therefore M^4 - 2M^3 - 48M^2 = 0$$

$$\therefore M^4 = 2M^3 + 48M^2$$

$$= 2(2M^2 + 48M) + 48M^2$$

$$= 52M^2 + 96M.$$

FP2 January 2008

3. $M = \begin{pmatrix} 7 & 3 \\ -4 & -1 \end{pmatrix}$

(i) use $\det(M - \lambda I) = 0$ for characteristic equation

$\det\begin{pmatrix} 7-\lambda & 3 \\ -4 & -1-\lambda \end{pmatrix} = 0 \Rightarrow -(7-\lambda)(1+\lambda) + 12 = 0$

$\therefore (1+\lambda)(7-\lambda) - 12 = 0$

$\therefore -\lambda^2 + 6\lambda + 7 - 12 = 0$

$\therefore \lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 5)(\lambda - 1) = 0$

$\therefore \lambda = 5$ or $\lambda = 1$

$\therefore$ eigenvalues are 1 and 5

for eigenvector corresponding to eigenvalue $\lambda = 1$

$\begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 6x + 3y = 0 \\ -4x - 2y = 0 \end{matrix} \Rightarrow \begin{matrix} x = 1 \\ y = -2 \end{matrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

for eigenvector corresponding to eigenvalue $\lambda = 5$

$\begin{pmatrix} 2 & 3 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 2x + 3y = 0 \\ -4x - 6y = 0 \end{matrix} \quad \begin{matrix} x = 3 \\ y = -2 \end{matrix} \Rightarrow \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(ii) $P^{-1}MP = D$ $\qquad M^n = S\Lambda^n S^{-1}$ where S eigenvector matrix, Λ eigenvalue matrix

$\therefore P = \begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

(iii) $M^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ using $M^n = PD^nP^{-1}$ (from above)

$M^n = \begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}^n P^{-1}$ where $P^{-1} = \dfrac{1}{\det P}\begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix}$

$\therefore M^n = \begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5^n \end{pmatrix}\dfrac{1}{4}\begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix} = \dfrac{1}{4}\begin{pmatrix} 1 & 3\times5^n \\ -2 & -2\times5^n \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix}$

$\therefore M^n = \dfrac{1}{4}\begin{pmatrix} -2 + 6\times5^n & -3 + 3\times5^n \\ 4 - 4\times5^n & +6 - 2\times5^n \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} + \frac{3}{2}\times5^n & -\frac{3}{4} + \frac{3}{4}\times5^n \\ 1 - 5^n & +\frac{3}{2} - \frac{1}{2}\times5^n \end{pmatrix}$

$a = -\frac{1}{2} + \frac{3}{2}\times5^n$; $b = \frac{-3}{4} + \frac{3}{4}\times5^n$; $c = 1 - 5^n$; $d = \frac{3}{2} - \frac{1}{2}\times5^n$.

FP2 June 2008.

3 (i) $Q = \begin{pmatrix} 2 & -1 & k \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ $k \neq 3$ $\det Q = 2\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$

$\therefore \det Q = 2(-1) + (-1) + k = k-3$

Cofactors $A_1 = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$ $A_2 = -\begin{vmatrix} -1 & k \\ 1 & 2 \end{vmatrix}$ $A_3 = \begin{vmatrix} -1 & k \\ 0 & 1 \end{vmatrix}$

$B_1 = -\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}$ $B_2 = \begin{vmatrix} 2 & k \\ 3 & 2 \end{vmatrix}$ $B_3 = -\begin{vmatrix} 2 & k \\ 1 & 1 \end{vmatrix}$

$C_1 = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$ $C_2 = -\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}$ $C_3 = \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}$

$$\therefore Q^{-1} = \frac{1}{\det Q}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \frac{1}{(k-3)}\begin{pmatrix} -1 & k+2 & -1 \\ 1 & 4-3k & k-2 \\ 1 & -5 & 1 \end{pmatrix}$$

When $k = 4$, $Q^{-1} = \begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix}$

(ii) M has eigenvectors $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ corresponding to eigenvalues 1, −1 and 3.

Want P and D such that $P^{-1}MP = D$

cf $M^n = S\Lambda^n S^{-1}$ where S is eigenvector matrix

$\therefore M = PDP^{-1}$ Λ is eigenvalue matrix

$$\therefore M = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix}$$ since $P^{-1} = Q^{-1}$ from (i)

$$\therefore M = \begin{pmatrix} 2 & 1 & 12 \\ 1 & 0 & 3 \\ 3 & -1 & 6 \end{pmatrix}\begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 11 & -56 & 12 \\ 2 & -9 & 2 \\ 2 & -4 & 1 \end{pmatrix}$$

(iii) Cayley Hamilton Theorem states M is also root of own Char. Eqn.

Characteristic Equation is $(\lambda-1)(\lambda+1)(\lambda-3) = 0$

$(\lambda+1)(\lambda^2 - 4\lambda + 3) = 0$ $\Rightarrow \lambda^3 - 4\lambda^2 + 3\lambda + \lambda^2 - 4\lambda + 3 = 0$

$\therefore\ \lambda^3 - 3\lambda^2 - \lambda + 3 = 0$

$\therefore\ M^3 - 3M^2 - M + 3I = 0$ $\qquad$ $M^3 = 3M^2 + M - 3I$

$\therefore\ M^4 - 3M^3 - M^2 + 3M = 0$

$\therefore\ M^4 = 3M^3 + M^2 - 3M$

$= 3(3M^2 + M - 3I) + M^2 - 3M$

$= 9M^2 + 3M - 9I + M^2 - 3M$

$\therefore\ M^4 = 10M^2 - 9I$ $\qquad$ cf $M^4 = aM^2 + bM + cI$

$a = 10 \quad b = 0 \quad c = -9$

FP2 January 2009.

3(b)(i) $M = \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix}$ To find the eigenvalues we need $\det(M-\lambda I)=1$

$\therefore (0.2-\lambda)(0.7-\lambda) - 0.24 = 0$

$\therefore \lambda^2 - 0.9\lambda + 0.14 - 0.24 = 0$

$\therefore \lambda^2 - 0.9\lambda - 0.1 = 0$

$(\lambda + 0.1)(\lambda - 1) = 0$

$\therefore \lambda = 1$ or $\lambda = -0.1$

The eigenvalues are 1 and -0.1

For $\lambda = 1$ $\begin{pmatrix} 0.2-1 & 0.8 \\ 0.3 & 0.7-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -0.8x + 0.8y = 0 \\ 0.3x - 0.3y = 0 \end{matrix}$

$\therefore x = y$ so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ could be an eigenvector for $\lambda = 1$

For $\lambda = -0.1$ $\begin{pmatrix} 0.2+0.1 & 0.8 \\ 0.3 & 0.7+0.1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 0.3x + 0.8y = 0 \\ 0.3x + 0.8y = 0 \end{matrix}$

$\therefore x = -\frac{8}{3}y$ so $\begin{pmatrix} 8 \\ -3 \end{pmatrix}$ could be an eigenvector for $\lambda = -0.1$

(ii) If $M = QDQ^{-1}$ Q will be matrix of eigenvectors

D will be matrix of eigenvalues

$\therefore Q = \begin{pmatrix} 1 & 8 \\ 1 & -3 \end{pmatrix}$ and $D \begin{pmatrix} 1 & 0 \\ 0 & -0.1 \end{pmatrix}$

FP2 June 2009

2(i) $M = \begin{pmatrix} 3 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ $(M-\lambda I) = \begin{pmatrix} 3-\lambda & 1 & -2 \\ 0 & -1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{pmatrix}$

For characteristic equation $\det(M-\lambda I)=0$

$\therefore -(3-\lambda)(1+\lambda)(1-\lambda) - 0 - 2(2(1+\lambda)) = 0$

$\therefore -(3-\lambda)(1-\lambda^2) - 4(1+\lambda) = 0$

$\therefore 3 - 3\lambda^2 - \lambda + \lambda^3 + 4 + 4\lambda = 0$

$\lambda^3 - 3\lambda^2 + 3\lambda + 7 = 0$ This is the char. equ.

Since $\det(M-\lambda I) = -\lambda^3 + 3\lambda^2 - 3\lambda - 7$ then when $\lambda = 0$

$\Rightarrow \operatorname{Det} M = -7$

(ii) If $\lambda = -1$ is an eigenvalue of M then the ch. eq will equal 0

$\therefore (-1)^3 - 3(-1)^2 + 3(-1) + 7 = -1 - 3 - 3 + 7 = 0$ so -1 is an eigenvalue

$\therefore (\lambda+1)(\lambda^2 - 4\lambda + 7) = 0$

$\lambda^2 - 4\lambda + 7 = 0 \Rightarrow (\lambda-2)^2 - 4 + 7 = 0$

$\therefore (\lambda-2)^2 = -3$ so no real roots here

$\therefore$ The other 2 roots are complex

If $\lambda = -1$ $\begin{pmatrix} 3+1 & 1 & -2 \\ 0 & -1+1 & 0 \\ 2 & 0 & 1+1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $4x + y - 2z = 0$
$2x + 2z = 0$

$\therefore x = -z$

$\therefore y = 2z - 4x = 2z - 4(-z) = 6z$

let $z = 1$ then the corresponding eigenvector is $\begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix}$

$3x + y - 2z = -0.1$
$-y = 0.6$
$2x + z = 0.1$
$\Rightarrow \begin{pmatrix} 3 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -0.1 \\ 0.6 \\ 0.1 \end{pmatrix}$

So from $(M-\lambda I)s = 0 \Rightarrow Ms = \lambda I s$

where s is an eigenvector corresponding to the eigenvalue λ

eg. $M\begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix} = (-1)(I)\begin{pmatrix} -1 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ -1 \end{pmatrix}$ so use $\begin{pmatrix} -0.1 \\ 0.6 \\ 0.1 \end{pmatrix}$ for s then

$M\begin{pmatrix} -0.1 \\ 0.6 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.1 \\ -0.6 \\ -0.1 \end{pmatrix}$ $\therefore x = 0.1,\ y = -0.6,\ z = -0.1$

(iii) Cayley Hamilton theorem states that M is a root of its own characteristic equation

Since $\lambda^3 - 3\lambda^2 + 3\lambda + 7 = 0$

$\therefore M^3 - 3M^2 + 3M + 7I = 0$

$\therefore M^3 = 3M^2 - 3M - 7I$

$\therefore M^2 = 3M - 3I - 7M^{-1}$

$\therefore 7M^{-1} = -M^2 + 3M - 3I$

$\therefore M^{-1} = -\frac{1}{7}M^2 + \frac{3}{7}M - \frac{3}{7}I$

(iv) If $M = \begin{pmatrix} 3 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ then $M^2 = \begin{pmatrix} 3 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$$\therefore M^2 = \begin{pmatrix} 9-4 & 3-1 & -6-2 \\ 0 & 1 & 0 \\ 6+2 & 2 & -4+1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & -8 \\ 0 & 1 & 0 \\ 8 & 2 & -3 \end{pmatrix}$$

$$\therefore M^{-1} = -\frac{1}{7}\begin{pmatrix} 5 & 2 & -8 \\ 0 & 1 & 0 \\ 8 & 2 & -3 \end{pmatrix} + \frac{3}{7}\begin{pmatrix} 3 & 1 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix} - \frac{3}{7}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore M^{-1} = \begin{pmatrix} \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \\ 0 & -1 & 0 \\ -\frac{2}{7} & -\frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

FP2. January 2010

3 (i) $M = \begin{pmatrix} 1 & 1 & a \\ 2 & -1 & 2 \\ 3 & -2 & 2 \end{pmatrix}$ $a \neq 4$ $\det M = 1\begin{vmatrix} -1 & 2 \\ -2 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} + a\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix}$

$\therefore \det M = (-2+4) - (4-6) + a(-4+3)$

$= 4-a$

$A_1 = \begin{vmatrix} -1 & 2 \\ -2 & 2 \end{vmatrix}$ $A_2 = -\begin{vmatrix} 1 & a \\ -2 & 2 \end{vmatrix}$ $A_3 = \begin{vmatrix} 1 & a \\ -1 & 2 \end{vmatrix}$

$B_1 = -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}$ $B_2 = \begin{vmatrix} 1 & a \\ 3 & 2 \end{vmatrix}$ $B_3 = -\begin{vmatrix} 1 & a \\ 2 & 2 \end{vmatrix}$

$C_1 = \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix}$ $C_2 = -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}$ $C_3 = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}$

$\therefore M^{-1} = \frac{1}{(4-a)}\begin{pmatrix} 2 & -2-2a & 2+a \\ 2 & 2-3a & 2a-2 \\ -1 & 5 & -3 \end{pmatrix}$

When $a = -1$

$\therefore M^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 0 & 1 \\ 2 & 5 & -4 \\ -1 & 5 & -3 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 2 \\ 3 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ b \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} -2 \\ b \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 0 & 1 \\ 2 & 5 & -4 \\ -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} -2 \\ b \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4+1 \\ -4+5b-4 \\ 2+5b-3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 \\ 5b-8 \\ -1+5b \end{pmatrix}$

$\therefore x = -\frac{3}{5}$; $y = b - \frac{8}{5}$; $z = b - \frac{1}{5}$

(iii) $\begin{pmatrix} 1 & 1 & 4 \\ 2 & -1 & 2 \\ 3 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ b \\ 1 \end{pmatrix}$ So using $a=4$ means $\det M = 0$.

① $x + y + 4z = -2$ ② $-$ 2① $\Rightarrow$ $-3y - 6z = b+4$ $\Rightarrow$ $y + 2z = -\frac{(b+4)}{3}$

② $2x - y + 2z = b$ ③ $-$ 3① $\Rightarrow$ $-5y - 10z = 7$ $\Rightarrow$ $y + 2z = -\frac{7}{5}$

③ $3x - 2y + 2z = 1$ Consistent if $\frac{b+4}{3} = \frac{7}{5}$ $\therefore b = \frac{1}{5}$

Two planes intersecting give a line.

FP1 June 2010.

3 (a) (i) Characteristic eq. $2\lambda^3 + \lambda^2 - 13\lambda + 6 = 0$ of a matrix M

If $\lambda = 2$ is an eigenvalue of M then $f(\lambda) = 0$ when $\lambda = 2$

$f(2) = 2(2)^3 + (2)^2 - 13(2) + 6$

$= 16 + 4 - 26 + 6$

$= 0$ $\therefore$ $\lambda = 2$ is an eigenvalue

$\therefore (\lambda - 2)(2\lambda^2 + 5\lambda - 3) = 0$

$\therefore (\lambda - 2)(2\lambda - 1)(\lambda + 3) = 0$

$\therefore \lambda = 2$, $\lambda = \frac{1}{2}$ or $\lambda = -3$, ie other eigenvalues are $\frac{1}{2}$ and -3

(ii) Eigenvector corresponding to $\lambda = 2$ is $\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$ $M^n = S\Lambda^n S^{-1}$

Using $MS = \lambda I S$ $\therefore M^n S = S\Lambda^n$

then $M\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 2 \end{pmatrix}$

and $M^2\begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \end{pmatrix} = \lambda^2\begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \end{pmatrix} = 4\begin{pmatrix} 1 \\ -1 \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ \frac{4}{3} \end{pmatrix}$

If $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$ since $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$

then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ \frac{1}{2} \end{pmatrix}$

(iii) $M^4 = AM^2 + BM + CI$ Comes from using Cayley Hamilton thm which states M is a root of its own characteristic eqn.

$\therefore 2M^3 + M^2 - 13M + 6I = 0$ $-M^3 = +\frac{1}{2}M^2 - \frac{13}{2}M + 3I$

$\therefore 2M^4 + M^3 - 13M^2 + 6M = 0$

$\therefore 2M^4 = -M^3 + 13M^2 - 6M$

$= \left(\frac{1}{2}M^2 - \frac{13}{2}M + 3I\right) + 13M^2 - 6M = \frac{27}{2}M^2 - \frac{25}{2}M + 3I$

$\therefore M^4 = \frac{27}{4}M^2 - \frac{25}{4}M + \frac{3}{2}I$

(b) overleaf

(b) 2×2 matrix N has eigenvalues -1 and 2, eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively.

Knowing the eigenvectors and eigenvalues we can use "$M^n = S\Lambda^n S^{-1}$"

$\therefore$ $N = S\Lambda S^{-1}$ where S is the eigenvector matrix and Λ is eigenvalue matrix

$\therefore$ $N = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} S^{-1}$ Inverse $S = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$

$\therefore$ $N = \begin{pmatrix} -1 & -2 \\ -2 & 2 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & -3 \\ -6 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$

FP2 January 2011

3 (i) $M = \begin{pmatrix} 1 & -4 & 5 \\ 2 & 3 & -2 \\ -1 & 4 & 1 \end{pmatrix}$ The characteristic equation comes from $\det(M - \lambda I) = 0$

$$(M - \lambda I) = \begin{pmatrix} 1-\lambda & -4 & 5 \\ 2 & 3-\lambda & -2 \\ -1 & 4 & 1-\lambda \end{pmatrix}$$

$$\begin{aligned} \therefore \det(M-\lambda I) &= (1-\lambda)\{(3-\lambda)(1-\lambda)+8\} + 4\{2(1-\lambda)-2\} + 5\{8+(3-\lambda)\} \\ &= (1-\lambda)(11+\lambda^2-4\lambda) - 8\lambda + 55 - 5\lambda \\ &= 11 + \lambda^2 - 4\lambda - 11\lambda - \lambda^3 + 4\lambda^2 + 55 - 13\lambda \\ &= -\lambda^3 + 5\lambda^2 - 28\lambda + 66 \end{aligned}$$

$\therefore$ Ch. Eq is $\lambda^3 - 5\lambda^2 + 28\lambda - 66 = 0$

(ii) If $\lambda = 3$ is a root and an eigenvalue of M then $f(3) = 0$

$$\therefore (3)^3 - 5(3)^2 + 28(3) - 66 = 27 - 45 + 84 - 66$$
$$= 0$$

$\therefore \lambda = 3$ is an eigenvalue of M.

$$\therefore \lambda^3 - 5\lambda^2 + 28\lambda - 66 \equiv (\lambda-3)(\lambda^2 - 2\lambda + 22)$$

$\lambda^2 - 2\lambda + 22$ has discriminant $4 - 88 < 0$

so no further real roots.

(iii) eigenvector of unit length corresponding to $\lambda = 3$

$$(M - \lambda I) \Rightarrow \begin{pmatrix} -2 & -4 & 5 \\ 2 & 0 & -2 \\ -1 & 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore -2x - 4y + 5z = 0$ $\Rightarrow -2z - 4y + 5z = 0$

$2x - 2z = 0$ $\Rightarrow x = z$ $\Rightarrow y = \frac{3}{4}z$

$-x + 4y - 2z = 0$

$\therefore$ An eigenvector could be $\begin{pmatrix} 1 \\ 3/4 \\ 1 \end{pmatrix}$ for $\lambda = 3$ (when $z = 1$ say)

Want v to be unit vector

So multiply by 4 to get integer elements and find modulus

$$\Rightarrow \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix} \text{ and } \left|\begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}\right| = \sqrt{4^2+3^2+4^2} = \sqrt{41} \quad \therefore v = \frac{1}{\sqrt{41}}\begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$$

(iii) continued. The magnitude of $M^n v$ is given by

$$M^n = S\Lambda^n S^{-1} \Rightarrow M^n S = S\Lambda^n$$

$$\therefore M^n v = v\,3^n$$

$\therefore$ magnitude is 3^n

(iv) Cayley Hamilton states M is a root of its own characteristic eqn.

$$\therefore \lambda^3 - 5\lambda^2 + 28\lambda - 66 = 0$$

$$\Rightarrow M^3 - 5M^2 + 28M - 66I = 0$$

$$\therefore M^2 - 5M + 28I - 66M^{-1} = 0$$

$$\therefore 66M^{-1} = M^2 - 5M + 28I$$

$$\therefore M^{-1} = \frac{1}{66}M^2 - \frac{5}{66}M + \frac{28}{66}I$$

FP2 June 2011

3(i) $M = \begin{pmatrix} 1 & -1 & k \\ 5 & 4 & 6 \\ 3 & 2 & 4 \end{pmatrix}$ M will not have an inverse if det M = 0

det $M = 1\begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} 5 & 6 \\ 3 & 4 \end{vmatrix} + k\begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix}$

$\therefore$ det $M = 16 - 12 + 20 - 18 + k(10-12)$

$= 6 + k(10-12)$

$\therefore \quad 2k = 6 \Rightarrow k = 3$ So M has inverse if $k \neq 3$.

$M^{-1} = \dfrac{1}{\det M}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$ where

$A_1 = \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix}$ $\quad A_2 = -\begin{vmatrix} -1 & k \\ 2 & 4 \end{vmatrix}$ $\quad A_3 = \begin{vmatrix} -1 & k \\ 4 & 6 \end{vmatrix}$

$B_1 = -\begin{vmatrix} 5 & 6 \\ 3 & 4 \end{vmatrix}$ $\quad B_2 = \begin{vmatrix} 1 & k \\ 3 & 4 \end{vmatrix}$ $\quad B_3 = -\begin{vmatrix} 1 & k \\ 5 & 6 \end{vmatrix}$

$C_1 = \begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix}$ $\quad C_2 = -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$ $\quad C_3 = \begin{vmatrix} 1 & -1 \\ 5 & 4 \end{vmatrix}$

$$\therefore M^{-1} = \frac{1}{6-2k}\begin{pmatrix} 4 & 4+2k & -6-4k \\ -2 & 4-3k & -6+5k \\ -2 & -5 & 9 \end{pmatrix}$$

(ii) If $k = 3$ $\quad M\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 4 & 6 \\ 3 & 2 & 4 \end{pmatrix}\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$

(iii) If $Ms = \lambda s$ for an eigenvalue λ and corresponding eigenvector s then if $s = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$ $\quad \lambda = 1$

(iv) ① $x - y + 3z = t$

② $5x + 4y + 6z = 1$ $\quad \Rightarrow \quad \begin{pmatrix} 1 & -1 & 3 \\ 5 & 4 & 6 \\ 3 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}$

③ $3x + 2y + 4z = 0$

3① − ③ $\quad -5y + 5z = 3t$

3② − 2③ $\quad 2y - 2z = 3$

$\left.\begin{array}{l} -y + z = \frac{3}{5}t \\ -y + z = -\frac{3}{2} \end{array}\right\}$ $\quad \frac{3}{5}t = -\frac{3}{2} \Rightarrow t = -\frac{5}{2}$

so the equations will have a solution if $t = -5/2$

The general solution when $t = -5/2$

$$z = y - \frac{3}{2} \qquad x - y + 3z = -\frac{5}{2} \Rightarrow x = -2y + 2$$

Gen. Solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2\lambda + 2 \\ \lambda \\ \lambda - 3/2 \end{pmatrix}$ for some λ.

Could also write the others in terms of x instead of y. etc.

The solution represents two intersecting planes ie. a straight line.

FP2 January 2012.

3 (i) $M = \begin{pmatrix} 3 & -1 & 2 \\ -4 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix}$ Characteristic eq. given by $\det(M-\lambda I)=0$

$$\det(M-\lambda I) = \begin{vmatrix} 3-\lambda & -1 & 2 \\ -4 & 3-\lambda & 2 \\ 2 & 1 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\{(3-\lambda)(-1-\lambda)-2\} + 1\{-4(-1-\lambda)-4\} + 2\{-4-2(3-\lambda)\}$$
$$= (3-\lambda)(-5+\lambda^2-2\lambda) + 4\lambda - 20 + 4\lambda$$
$$= -15 + 3\lambda^2 - 6\lambda + 5\lambda - \lambda^3 + 2\lambda^2 + 4\lambda - 20 + 4\lambda$$
$$= -\lambda^3 + 5\lambda^2 + 7\lambda - 35$$

If $\det(M-\lambda I)=0 \Rightarrow \lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$ is the C.E.

(ii) If $\lambda = 5$ is an eigenvalue of M then $f(\lambda)=0$

$\therefore (5)^3 - 5(5)^2 - 7(5) + 35 = 125 - 125 - 35 + 35$
$= 0$

$\therefore \lambda = 5$ is an eigenvalue of M.

$\therefore (\lambda-5)(\lambda^2 \quad -7) = 0$

$\therefore \lambda^2 - 7 = 0 \Rightarrow \lambda = \pm\sqrt{7}$

The eigenvalues are $5, -\sqrt{7}, +\sqrt{7}$

(iii) unit vector v corresponding to $\lambda=5$ $\quad M-\lambda I$ where $\lambda=5$

$$\begin{pmatrix} -2 & -1 & 2 \\ -4 & -2 & 2 \\ 2 & 1 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{ll} -2x - y + 2z = 0 & ① \\ -4x - 2y + 2z = 0 & ② \\ 2x + y - 6z = 0 & ③ \end{array}$$

① + ③ $-4z = 0 \Rightarrow z = 0 \quad \therefore 4x = -2y \Rightarrow -2x = y$

If $x = 1$ then an eigenvector could be $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

A unit vector? $\sqrt{1^2+(-2)^2} = \sqrt{5}$

$\therefore v = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

$M^2 v = \lambda^2 v$ magnitude 25 in the direction of v

$M^{-1} v = \lambda^{-1} v$ magnitude $\frac{1}{5}$ in the direction of v.

(iv) Cayley Hamilton Thm states M is a root of its own char. equ.

$$\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$$

$$\therefore \quad M^3 - 5M^2 - 7M + 35I = 0$$

$$\therefore \quad M^3 = 5M^2 + 7M - 35I$$

$$M^4 = 5M^3 + 7M^2 - 35M$$

$$= 5(5M^2 + 7M - 35I) + 7M^2 - 35M$$

$$= 25M^2 + 35M - 175I + 7M^2 - 35M$$

$$= 32M^2 - 175I$$

$$\therefore \quad a = 32 \quad b = 0 \quad c = -175$$