FP2. Maclaurin Series. June 2006. (and other series, integrating trig)

1.(b) (i) $f(x) = \tan\left(\frac{\pi}{4} + x\right)$ to x^2.

$\therefore f'(x) = \sec^2\left(\frac{\pi}{4} + x\right)$ $\qquad f''(x) = 0 + 2\tan\left(\frac{\pi}{4} + x\right)\sec^2\left(\frac{\pi}{4} + x\right)$

$\therefore f'(x) = 1 + \tan^2\left(\frac{\pi}{4} + x\right)$

$\therefore f'(x) = \sec^2\left(\frac{\pi}{4} + x\right)$ $\qquad f''(x) = 2\tan\left(\frac{\pi}{4} + x\right)\sec^2\left(\frac{\pi}{4} + x\right)$

Maclaurin Series $\quad f(x) \approx f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$

$\therefore f(0) = \tan\left(\frac{\pi}{4}\right) = 1$

$f'(0) = \sec^2\left(\frac{\pi}{4}\right) = 2$

$f''(0) = 2\tan\left(\frac{\pi}{4}\right)\sec^2\left(\frac{\pi}{4}\right) = 4$

$\therefore f(x) \approx 1 + 2x + \frac{4x^2}{2} \ldots = 1 + 2x + 2x^2 \ldots$

(ii) When h is small $\quad \int_{-h}^{h} x^2 \tan\left(\frac{\pi}{4} + x\right) dx \approx \frac{2}{3}h^3 + \frac{4}{5}h^5$

$$\int_{-h}^{h} x^2 \tan\left(\frac{\pi}{4} + x\right) dx \approx \int_{-h}^{h} x^2 (1 + 2x + 2x^2)\, dx \approx \int_{-h}^{h} (x^2 + 2x^3 + 2x^4)\, dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^4}{4} + \frac{2x^5}{5}\right]_{-h}^{h}$$

$$= \left[\frac{h^3}{3} + \frac{h^4}{2} + \frac{2h^5}{5}\right] - \left[\frac{(-h)^3}{3} + \frac{(-h)^4}{2} + \frac{2(-h)^5}{5}\right]$$

$$= \frac{2h^3}{3} + \frac{4h^5}{5}$$

FP2. January 2007.

1(c)(i) Maclaurin Series for $\tan x$ to x^3

$f(x) = \tan x$ $\qquad f(0) = 0$

$f'(x) = \sec^2 x$ $\qquad f'(0) = 1$

$f''(x) = 2\sec x\,(\sec x \tan x)$ $\qquad f''(0) = 0$

$= 2\sec^2 x \tan x$ $\qquad f'''(0) = 2$

$f'''(x) = 4\sec x\,(\sec x \tan^2 x) + 2\sec^2 x\,(\sec^2 x)$

$= 4\sec^2 x \tan^2 x + 2\sec^4 x$

$$f(x) \approx f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) \ldots$$

$$\therefore f(x) \approx 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}.2 \ldots$$

$$\therefore f(x) \approx x + \frac{2x^3}{3!} \ldots = x + \frac{x^3}{3} + \ldots$$

(ii) $$\int_h^{4h} \frac{\tan x}{x}\,dx \cong \int_h^{4h} \frac{1}{x}\left(x + \frac{x^3}{3} + ..\right) dx \approx \int_h^{4h} \left(1 + \frac{x^2}{3} + ..\right) dx$$

$$\approx \left[x + \frac{x^3}{9}\right]_h^{4h}$$

$$\approx \left[(4h) + \frac{(4h)^3}{9}\right] - \left[h + \frac{h^3}{9}\right]$$

$$\approx 3h + \frac{64h^3}{9} - \frac{h^3}{9} \qquad \frac{63}{9} = 7$$

$$\approx 3h + 7h^3.$$

FP2 June 2007

1(c) (i) $f(x) = \arccos 2x$

$f'(x) = \left(\frac{-2}{\sqrt{1-(2x)^2}}\right)$

$f'(x) = \frac{-2}{\sqrt{1-4x^2}}$

let $f(u) = \arccos u \quad (u=2x)$

$f'(u) = \frac{-1}{\sqrt{1-u^2}}$

$\therefore f'(x) = f'(u) \times \frac{du}{dx} = 2f'(u)$

(ii) $f'(x) = -2(1-4x^2)^{-1/2}$

$= -2\left\{1 + (-\tfrac{1}{2})(-4x^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})(-4x^2)^2}{2!} + \ldots\right\}$

$= -2\{1 + 2x^2 + 6x^4 + \ldots\}$

$= -2 - 4x^2 - 12x^4 + \ldots$

Integrate with respect to x

$\therefore f(x) = -2x - \frac{4x^3}{3} - \frac{12x^5}{5} + c \ldots$

Since $f(x) = \arccos 2x$

$\therefore f(0) = \arccos(0) = \frac{\pi}{2}$

$\therefore \frac{\pi}{2} = c$

$\therefore f(x) = -2x - \frac{4x^3}{3} - \frac{12x^5}{5} + \frac{\pi}{2}$

FP2 January 2008

1 (b) (i) $f(x) = \arctan(\sqrt{3}+x)$ find $f'(x)$ and $f''(x)$

$$\therefore f'(x) = \frac{1}{1+(\sqrt{3}+x)^2} = \left(1+(\sqrt{3}+x)^2\right)^{-1} \qquad \left(\frac{1}{1+x^2}\right)$$

$$\therefore f''(x) = -1\left(1+(\sqrt{3}+x)^2\right)^{-2} \times 2(\sqrt{3}+x)$$

$$= \frac{-2(\sqrt{3}+x)}{\left(1+(\sqrt{3}+x)^2\right)^2}$$

(ii) Maclaurin series to x^2

$$f(x) \approx f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

$$f(0) = \arctan(\sqrt{3}) = \frac{\pi}{3} \qquad \tan(\pi/3) = \sqrt{3}$$

$$f'(0) = \frac{1}{1+(\sqrt{3})^2} = \frac{1}{4}$$

$$f''(0) = \frac{-2\sqrt{3}}{\left(1+(\sqrt{3})^2\right)^2} = \frac{-2\sqrt{3}}{16}$$

$$\therefore f(x) = \frac{\pi}{3} + \frac{x}{4} + \left(\frac{x^2}{2}\right)\left(\frac{-2\sqrt{3}}{16}\right)$$

$$\therefore f(x) = \frac{\pi}{3} + \frac{x}{4} - \frac{\sqrt{3}}{16}x^2 + \dots$$

(iii) $$\int_{-h}^{h} x \arctan(\sqrt{3}+x)\,dx \approx \frac{1}{6}h^3$$

$$\therefore \int_{-h}^{h} x\left\{\frac{\pi}{3} + \frac{x}{4} - \frac{\sqrt{3}}{16}x^2 \dots\right\}dx = \int_{-h}^{h}\left\{\frac{\pi x}{3} + \frac{x^2}{4} - \frac{\sqrt{3}}{16}x^3\right\}dx$$

$$\cong \left[\frac{\pi x^2}{6} + \frac{x^3}{12} - \frac{\sqrt{3}}{64}x^4\right]_{-h}^{h}$$

$$\cong \left[\frac{\pi h^2}{6} + \frac{h^3}{12} - \frac{\sqrt{3}}{64}h^4\right] - \left[\frac{\pi}{6}h^2 - \frac{h^3}{12} - \frac{\sqrt{3}}{64}h^4\right]$$

$$\cong \frac{h^3}{6}$$

FP2 June 2008

1 (b) $\displaystyle\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx$ $\qquad\qquad$ $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)+c$

$$= \int_0^1 \frac{1}{\sqrt{3\left(\frac{4}{3}-x^2\right)}}\,dx = \int_0^1 \frac{1}{\sqrt{3}}\frac{1}{\sqrt{\left(\frac{4}{3}-x^2\right)}}\,dx$$

$$= \left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{x}{\sqrt{\frac{4}{3}}}\right)\right]_0^1 = \left[\frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}x}{2}\right)\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{\sqrt{3}}\arcsin(0)$$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\sin 0 = 0.$

$$= \frac{1}{\sqrt{3}}\cdot\frac{\pi}{3}$$

(c) (i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$
$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\dots\right) - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}\dots\right]$$
$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots$$

(ii) $\displaystyle\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = \ln 3$

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = 1 + \frac{1}{3\times 4} + \frac{1}{5\times 16} + \frac{1}{7\times 64} + \dots$$

cf $2x + \frac{2x^3}{3} + \frac{2x^5}{5}$ use $x = \frac{1}{2}$ $\quad 2\times\left(\frac{1}{2}\right) + \frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{2}{5}\left(\frac{1}{2}\right)^5$

$$= 1 + \frac{1}{3\times 4} + \frac{1}{5\times 16} \dots$$

$$\therefore \sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = \ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) = \ln 3.$$

FP2 January 2009.

1(a)(i)

$$f(x) = \cos(x) \qquad f(0) = 1$$
$$f'(x) = -\sin(x) \qquad f'(0) = 0$$
$$f''(x) = -\cos(x) \qquad f''(0) = -1$$
$$f'''(x) = \sin(x) \qquad f'''(0) = 0$$
$$f^{iv}(x) = \cos(x) \qquad f^{iv}(0) = 1$$

$$\therefore f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0)$$

$$\therefore f(x) \approx 1 + x.0 + \frac{x^2}{2}.(-1) + \frac{x^3}{6}.0 + \frac{x^4}{24}.1$$

$$\therefore f(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

(ii)

$$f(x) = \sec(x)$$
$$\approx 1 + ax^2 + bx^4$$

$x \to 0$ show $(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4)(1 + ax^2 + bx^4) \approx 1$

Since $f(x) = \cos(x)$ is reciprocal of $f(x) = \sec(x)$

$$\cos(x) \cdot \sec(x) = \cos(x) \cdot \frac{1}{\cos(x)} = 1$$

So $1 + ax^2 + bx^4 - \frac{1}{2}x^2 - \frac{1}{2}ax^4 - \frac{1}{2}bx^6 + \frac{1}{24}x^4 + \frac{1}{24}x^6 + \frac{1}{24}bx^8 =$

$$\therefore x^2\left(a - \frac{1}{2}\right) + x^4\left(b - \frac{1}{2}a + \frac{1}{24}\right) = 0$$ using $x^6 \to 0$, $x^8 \to 0$

$$\therefore a = \frac{1}{2} \quad \text{and} \quad b - \frac{1}{4} + \frac{1}{24} = 0$$
$$\therefore b = \frac{5}{24}$$

(b)(i) $y = \arctan\left(\frac{x}{a}\right)$

$$\therefore \frac{x}{a} = \tan y$$
$$\therefore x = a\tan y$$
$$\therefore \frac{dx}{dy} = a\sec^2 x$$

this assumes you already know it:

$y = \arctan(u)$

$\frac{dy}{du} = \frac{1}{1+u^2}$ but $u = \frac{x}{a}$, $\frac{du}{dx} = \frac{1}{a}$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a}$$
$$= \frac{a^2}{a^2+x^2}\cdot\frac{1}{a} = \frac{a}{a^2+x^2}$$

$$\therefore \frac{dx}{dy} = a(1+\tan^2 y) = a\left(1+\left(\frac{x}{a}\right)^2\right) = \frac{1}{a}(a^2+x^2)$$

$$\therefore \frac{dy}{dx} = \frac{a}{(a^2+x^2)}$$

(ii) (A)
$$\int_{-2}^{2} \frac{1}{(4+x^2)}\,dx = \frac{1}{2}\int_{-2}^{2}\frac{2}{(2^2+x^2)}$$
$$= \left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_{-2}^{2}$$
$$= \frac{1}{2}\arctan\left(\frac{2}{2}\right) - \frac{1}{2}\arctan\left(-\frac{2}{2}\right)$$
$$= \frac{\pi}{8} - \left(-\frac{\pi}{8}\right)$$
$$= \frac{\pi}{4}$$

(B)
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{4}{1+4x^2}\,dx = \int_{-1/2}^{1/2} 2\left(\frac{2}{1+4x^2}\right)dx$$

$$\left(\frac{x}{1/2}\right)^2 = 4x^2$$

$$y = \arctan(2x) \qquad a = 1/2$$

$$\frac{dy}{dx} = \frac{1/2}{\frac{1}{4}+x^2} = \frac{\frac{1}{2}\times 4}{1+4x^2}$$

$$= \Big[2\arctan(2x)\Big]_{-1/2}^{1/2}$$

$$= 2\arctan(1) - 2\arctan(-1)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi.$$

FP2 June 2009.

1 (a) (i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots \quad (-1 < x \le 1)$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \dots \quad (-1 \le x < 1)$

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} \dots \quad |x| < 1$

(ii) $\frac{1+x}{1-x} = 3 \Rightarrow 1+x = 3(1-x)$

$\therefore \quad 4x = 2$

$x = 1/2$

$\therefore \ln(3) = \frac{2 \cdot 1}{2} + \frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{2}{5}\left(\frac{1}{2}\right)^5 = 1.09583\dots$

$\frac{2}{7}\left(\frac{1}{2}\right)^7 = 0.002232\dots$

$\frac{2}{9}\left(\frac{1}{2}\right)^9 = 0.000484\dots$

$\frac{2}{11}\left(\frac{1}{2}\right)^{11} = 0.00008\dots$

$\therefore \ln(3) \sim 1.096$ (if 3 terms) ; 1.096 (4 terms) ; 1.097 (5 terms)

FP2 January 2010.

1 (a) $y = \arctan\sqrt{x}$ let $u = \sqrt{x} = x^{1/2}$ $\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$\therefore \frac{dy}{dx} = \frac{1}{(1+u^2)} \times \frac{1}{2\sqrt{x}}$

$= \frac{1}{2\sqrt{x}(1+x)}$

$\int_0^1 \frac{1}{\sqrt{x}(x+1)}\,dx = 2\int_0^1 \frac{1}{2\sqrt{x}(x+1)}\,dx$

$= 2\left[\arctan\sqrt{x}\right]_0^1$

$= 2\arctan 1 - 2\arctan 0$

$= 2\left(\frac{\pi}{4}\right) - 0$

$= \frac{\pi}{2}$

2 (c) Maclaurin's series for e^t

$e^t \approx 1 + t + \frac{t^2}{2!} + \ldots$

$e^t - 1 \approx t + \frac{t^2}{2} \approx t\left(1 + \frac{t}{2}\right)$

$\frac{t}{e^t-1} \approx \frac{t}{t+\frac{t^2}{2}}$

$\therefore \frac{t}{e^t-1} \approx \frac{1}{1+\frac{t}{2}}$

but $\left(1+\frac{t}{2}\right)^{-1} = 1 + (-1)\frac{t}{2} + (-1)(-2)\frac{t^2}{4} \ldots$

$= 1 - \frac{t}{2} + \frac{t^2}{2}$

$\therefore \frac{t}{e^t-1} \approx 1 - t/2$ when t close to 0 $t^2 \sim 0$.

FP2 June 2010.

1 (a)(i) $f(t) = \arcsin t$

$f'(t) = \frac{1}{\sqrt{1-t^2}} = (1-t^2)^{-1/2}$

$f''(t) = -\frac{1}{2}(1-t^2)^{-3/2}(-2t)$

$\therefore f''(t) = \frac{t}{(1-t^2)^{3/2}}$

(ii) Maclaurin Expansion for $\arcsin(x+\frac{1}{2})$

$\therefore f(x) \approx f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$ $\quad t = (0+\frac{1}{2})$ for $f'(0)$ etc

$= \arcsin(0+\frac{1}{2}) + x(1-(\frac{1}{2})^2)^{-1/2} + \frac{x^2}{2}\left(\frac{(\frac{1}{2})}{(1-(\frac{1}{2})^2)^{3/2}}\right) \ldots$

$= \arcsin(\frac{1}{2}) + \frac{\sqrt{4}x}{\sqrt{3}} + \frac{x^2}{2} \cdot \frac{4}{3\sqrt{3}} \ldots$ $\quad \frac{\frac{1}{2}}{(\frac{3}{4})^{3/2}} = \frac{4}{3\sqrt{3}}$

$= \frac{\pi}{6} + \frac{2x}{\sqrt{3}} + \frac{2x^2}{3\sqrt{3}}$

term in x^2 is therefore $\frac{2x^2}{3\sqrt{3}}$

FP2 January 2011.

1(b)(i) $f(x) = \arctan\left(\frac{x}{2}\right)$

$y = \arctan(x)$ $\quad u = \frac{x}{2}$

$\frac{dy}{dx} = \frac{1}{1+x^2}$ $\quad \frac{du}{dx} = \frac{1}{2}$

$y = \arctan(u)$

$\frac{dy}{dx} = \frac{1}{1+u^2} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{1+\left(\frac{x}{2}\right)^2}$

$$\therefore f'(x) = \frac{1}{1+\left(\frac{x}{2}\right)^2} \times \frac{1}{2}$$

$$= \frac{4}{4+x^2} \times \frac{1}{2}$$

$$= \frac{2}{4+x^2}$$

(ii) $f'(x) = 2(4+x^2)^{-1} = \frac{1}{2}\left(1+\left(\frac{x}{2}\right)^2\right)^{-1}$

$$= \frac{1}{2}\left\{1 + (-1)\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)}{2!}\left(\left(\frac{x}{2}\right)^2\right)^2 + \dots\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{x^2}{4} + \frac{x^4}{16} \dots\right\}$$

$$\therefore f'(x) = \frac{1}{2} - \frac{x^2}{8} + \frac{x^4}{32} \dots$$

$$\therefore f(x) = \int \left\{\frac{1}{2} - \frac{x^2}{8} + \frac{x^4}{32} \dots\right\} dx$$

$$= \frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{160} \dots + c$$

FP2 June 2011

1 (b) (i) $\displaystyle\int_{-1/2}^{1/2} \frac{1}{1+4x^2}\,dx$

$y = \arctan x$ $\quad \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

$y = \arctan\left(\frac{x}{a}\right)$ $\quad \dfrac{dy}{dx} = \dfrac{a}{a^2+x^2}$

$\left[\ y = \arctan(2x) \quad \dfrac{dy}{dx} = \dfrac{2}{4^2+x^2}\ \right]$

$a = \frac{1}{2}$ $\quad \dfrac{dy}{dx} = \dfrac{\frac{1}{2}}{\frac{1}{4}+x^2} = \dfrac{1}{2}\left(\dfrac{1}{4}+x^2\right)^{-1}$

$$\frac{1}{1+4x^2} = \frac{1}{4\left(\frac{1}{4}+x^2\right)}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$$

$$= \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\left(\frac{1}{4}+x^2\right)}\,dx$$

$$= \frac{1}{4}\Big[2\arctan(2x)\Big]_{-1/2}^{1/2}$$

$$= \frac{1}{4}\Big[2\arctan(1) - 2\arctan(-1)\Big]$$

$$= \frac{\pi}{4}$$

(ii) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1+4x^2)^{3/2}}\,dx$

only identity with $1+\ldots^2$ is
$1+\tan^2\theta = \sec^2\theta$

let $4x^2 = \tan^2\theta$ ie $x = \frac{1}{2}\tan\theta$

$\therefore \dfrac{dx}{d\theta} = \dfrac{1}{2}\sec^2\theta$

$$\int \frac{1}{(1+4x^2)^{3/2}}\,dx = \int \frac{1}{(1+\tan^2\theta)^{3/2}} \times dx$$

$$= \int \frac{1}{(\sec^2\theta)^{3/2}} \times \frac{1}{2}\sec^2\theta\,d\theta$$

$$= \int \frac{1}{\sec^3\theta} \times \frac{1}{2}\sec^2\theta\,d\theta$$

$$= \int \frac{1}{2\sec\theta}\,d\theta = \frac{1}{2}\int \cos\theta\,d\theta = \frac{1}{2}\sin\theta$$

When $x = \frac{1}{2}$ $\quad \theta = \tan^{-1}(1) = \pi/4$

When $x = -\frac{1}{2}$ $\quad \theta = \tan^{-1}(-1) = -\pi/4$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1+4x^2)^{3/2}}\,dx = \left[\frac{1}{2}\sin\theta\right]_{-\pi/4}^{\pi/4} = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)\right) = \frac{1}{\sqrt{2}}$$

FP2 January 2012

1 (b) $\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \Rightarrow \sin(x) \approx x - \frac{x^3}{6}$

$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \Rightarrow \cos(x) \approx 1 - \frac{x^2}{2}$

$\therefore \tan(x) = \frac{\sin(x)}{\cos(x)} \approx \frac{\left(x - \frac{x^3}{6}\right)}{\left(1 - \frac{x^2}{2}\right)}$

$$\left(1 - \frac{x^2}{2}\right)^{-1} = 1 + (-1)\left(-\frac{x^2}{2}\right) + \frac{(-1)(-2)}{2!}\left(-\frac{x^2}{2}\right)^2 \ldots$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{4} \ldots$$

$$\therefore \left(x - \frac{x^3}{6}\ldots\right)\left(1 + \frac{x^2}{2}\ldots\right) \approx x + \frac{x^3}{2} - \frac{x^3}{6} - \frac{x^5}{12} \ldots$$

$\therefore \tan(x) \approx x + \frac{x^3}{3}$ $\quad \therefore a = 1, \quad b = \frac{1}{3}$

(c) $$\int_0^1 \frac{1}{\sqrt{1 - \frac{1}{4}x^2}}\,dx$$

$$\frac{1}{\sqrt{1-(1/4)x^2}} = \frac{1}{\sqrt{\frac{4-x^2}{4}}} = \frac{2}{\sqrt{4-x^2}}$$

$$= \int_0^1 \frac{2}{\sqrt{4-x^2}}\,dx$$

$$= \left[2\arcsin\left(\frac{x}{2}\right)\right]_0^1$$

$$= 2\left(\frac{\pi}{6}\right) - 2(0)$$

$$= \frac{\pi}{3}$$