FP2 Hyperbolic functions.

4(a) $\sinh x + 4\cosh x = 8$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\therefore \frac{1}{2}\left(e^x - e^{-x}\right) + 4 \cdot \frac{1}{2} \cdot \left(e^x + e^{-x}\right) = 8$

$\therefore e^x - e^{-x} + 4e^x + 4e^{-x} = 16$

$\therefore e^{2x} - 1 + 4e^{2x} + 4 - 16e^x = 0$

$\therefore 5e^{2x} - 16e^x + 3 = 0$

$(5e^x - 1)(e^x - 3) = 0$

$\therefore e^x = \frac{1}{5}$ or $e^x = 3$

$\therefore \ln e^x = \ln\left(\frac{1}{5}\right)$ or $\ln e^x = \ln 3$

$\therefore x = \ln\left(\frac{1}{5}\right) = -\ln 5$

or $x = \ln 3$.

(b) $\displaystyle\int_0^2 e^x \sinh x \, dx = \int_0^2 e^x \cdot \frac{1}{2}\left(e^x - e^{-x}\right) dx$

$= \displaystyle\int_0^2 \left(\frac{1}{2}e^{2x} - \frac{1}{2}\right) dx$

$\dfrac{d}{dx}\left(e^{2x}\right) = 2e^{2x}$

$= \left[\frac{1}{4}e^{2x} - \frac{1}{2}x\right]_0^2$

$= \left(\frac{1}{4}e^4 - 1\right) - \left(\frac{1}{4}e^0\right)$

$= \frac{1}{4}e^4 - \frac{5}{4}$

(c) (i) let $y = \operatorname{arcsinh}\left(\frac{2}{3}x\right)$

$\frac{2}{3}x = \sinh y$

$x = \frac{3}{2}\sinh y$

$\dfrac{dx}{dy} = \dfrac{3}{2}\cosh y$

$\cosh^2 y = 1 + \sinh^2 y$

$= \frac{3}{2}\sqrt{1 + \sinh^2 y} = \frac{3}{2}\sqrt{1 + \left(\frac{2}{3}x\right)^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{\frac{2}{3}}{\sqrt{1 + \left(\frac{2x}{3}\right)^2}} = \dfrac{2}{\sqrt{9 + 4x^2}}$

(ii) Show $\int_0^2 \text{arsinh}\left(\frac{2}{3}x\right) dx = 2\ln 3 - 1.$

product rule rev.
$\int u\,dv = uv - \int v\,du$

$u = \text{arsinh}\left(\frac{2}{3}x\right) \Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{9+4x^2}}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$\therefore \int_0^2 \text{arsinh}\left(\frac{2}{3}x\right) dx = \left[x\,\text{arsinh}\left(\frac{2}{3}x\right)\right]_0^2 - \int_0^2 \frac{2x}{\sqrt{9+4x^2}}\,dx$

$\left[x\,\text{arsinh}\left(\frac{2x}{3}\right)\right]_0^2 = 2\,\text{arsinh}\left(\frac{4}{3}\right)$

$\frac{d}{dx}\left((9+4x^2)^{1/2}\right) = \frac{1}{2}(9+4x^2)^{-1/2} \times 8x = \frac{4x}{\sqrt{9+4x^2}}$

$\therefore \int_0^2 \frac{2x}{\sqrt{9+4x^2}}\,dx = \frac{1}{2}\left[(9+4x^2)^{1/2}\right]_0^2$

$= \frac{1}{2}(5-3) = 1$

$\therefore \int_0^2 \text{arsinh}\left(\frac{2x}{3}\right) dx = 2\,\text{arsinh}\left(\frac{4}{3}\right) - 1$

However $\text{arsinh}(x) = \ln\left\{x + \sqrt{x^2+1}\right\}$ (booklet)

So $\text{arsinh}\left(\frac{4}{3}\right) = \ln\left\{\frac{4}{3} + \sqrt{\left(\frac{4}{3}\right)^2+1}\right\}$

$= \ln\left\{\frac{4}{3} + \sqrt{\frac{16}{9}+1}\right\}$

$= \ln\left\{\frac{4}{3} + \sqrt{\frac{25}{9}}\right\}$

$= \ln\left\{\frac{4}{3} + \frac{5}{3}\right\}$

$= \ln\{3\}$

$\therefore \int_0^2 \text{arsinh}\left(\frac{2x}{3}\right) dx = 2\ln 3 - 1.$

FP2 June 2006.

4 (i) Prove $1+2\sinh^2 x = \cosh 2x$

$$1+2\sinh^2 x = 1+2\left\{\frac{1}{(2)^2}(e^x-e^{-x})^2\right\}$$
$$= 1+2\left\{\frac{1}{4}(e^{2x}+e^{-2x}-2)\right\}$$
$$= 1+\frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}-1$$
$$= \frac{1}{2}e^{2x}+\frac{1}{2}e^{-2x}$$
$$= \frac{1}{2}\left\{e^{2x}+e^{-2x}\right\}$$
$$= \cosh(2x)$$

(ii) If $2\cosh 2x + \sinh x = 5$

$\therefore\ 2(1+2\sinh^2 x)+\sinh x = 5$

$\therefore\ 4\sinh^2 x + \sinh x - 3 = 0$

$\therefore\ (4\sinh x - 3)(\sinh x + 1) = 0$

$\therefore\ \sinh x = \frac{3}{4}$ or $\sinh x = -1$

$$\therefore\ x = \operatorname{arsinh}\left(\tfrac{3}{4}\right) = \ln\left\{\frac{3}{4}+\sqrt{\left(\frac{3}{4}\right)^2+1}\right\}$$
$$= \ln\left\{\frac{3}{4}+\sqrt{\frac{25}{16}}\right\}$$
$$= \ln 2$$

$$\text{or}\quad x = \operatorname{arsinh}(-1) = \ln\left\{-1+\sqrt{(-1)^2+1}\right\}$$
$$= \ln\left\{-1+\sqrt{2}\right\}$$

$\therefore\ x = \ln 2$ or $x = \ln(\sqrt{2}-1)$

(iii) $$\int_0^{\ln 3} \sinh^2 x \, dx = \frac{10}{9} - \frac{1}{2}\ln 3$$

$1+2\sinh^2 x = \cosh 2x$

$\therefore\ \sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$$\therefore \int_0^{\ln 3} \frac{1}{2}(\cosh 2x - 1)\,dx = \left[\frac{1}{4}\sinh 2x - \frac{x}{2}\right]_0^{\ln 3}$$
$$= \left(\frac{1}{4}\sinh(2\ln 3) - \frac{1}{2}\ln 3\right) - (0)$$

But $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$

$$\therefore\ \frac{1}{4}\sinh(2\ln 3) = \frac{1}{2}\cdot\frac{1}{4}\cdot\left(e^{2\ln 3} - e^{-2\ln 3}\right) = \frac{1}{8}\left(e^{\ln 9} - e^{\ln\frac{1}{9}}\right)$$

$$\therefore \int_0^{\ln 3} \sinh^2 x \, dx = \frac{1}{8}\left(9 - \frac{1}{9}\right) - \frac{1}{2}\ln 3$$

$$= \frac{10}{9} - \frac{1}{2}\ln 3$$

(iv) $\displaystyle\int_3^5 \sqrt{x^2-9}\, dx$

$$\int \frac{1}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$$
$$= \ln\left\{x + \sqrt{x^2-a^2}\right\}$$

Having used $\int \sinh^2 x \, dx$ above

express $\int_3^5 \sqrt{x^2-9}\, dx$ using subst. to get to part iii.

If $x^2 = \cosh^2 u$ say $\sqrt{\cosh^2 u - 9}$ not quite $\sinh^2 u$

So $x^2 = (3\cosh u)^2 \Rightarrow \sqrt{9\cosh^2 u - 9} = \sqrt{9(\cosh^2 u - 1)}$

$= 3\sinh u.$

when $x = 3$ $3\cosh u = 3 \Rightarrow \cosh u = 1 \Rightarrow u = 0$

$x = 5$ $3\cosh u = 5 \Rightarrow \cosh u = \frac{5}{3} \Rightarrow u = \operatorname{arcosh}(5/3)$

and $\operatorname{arcosh}\left(\frac{5}{3}\right) = \ln\left\{\frac{5}{3} + \sqrt{\left(\frac{5}{3}\right)^2 - 1}\right\} = \ln\{3\}$

$$\therefore \int_3^5 \sqrt{x^2-9}\, dx = \int_0^{\ln 3} 3\sinh u \, dx \quad \text{but} \quad \frac{dx}{du} = 3\sinh u$$

$$= \int_0^{\ln 3} 3\sinh u \cdot 3\sinh u \, . du$$

$$= \int_0^{\ln 3} 9\sinh^2 u \, du$$

$$= 9\left\{\frac{10}{9} - \frac{1}{2}\ln 3\right\} \quad \text{from (iii)}$$

$$= 10 - \frac{9}{2}\ln 3.$$

FP2 January 2007.

4 (i) Show $\operatorname{arcosh} x = \ln\{x + \sqrt{x^2-1}\}$

let $y = \operatorname{arcosh} x$

$\therefore x = \cosh y = \dfrac{e^y + e^{-y}}{2}$

If $2x = e^y + e^{-y}$

$\Rightarrow e^{2y} + 1 - 2xe^y = 0$

$\therefore e^y = \dfrac{+2x \pm \sqrt{(-2x)^2 - 4.1.1}}{2.1} = \dfrac{2x \pm \sqrt{4x^2-4}}{2}$

$\therefore e^y = \dfrac{2x}{2} \pm \dfrac{2}{2}\sqrt{x^2-1} = x \pm \sqrt{x^2-1}$

However $y \geqslant 0$ $e^y \geqslant 1$ $\therefore e^y = x + \sqrt{x^2-1}$

$\therefore \operatorname{arcosh} x = y = \ln\{x + \sqrt{x^2-1}\}$

(ii) $\displaystyle\int_{2.5}^{3.9} \frac{1}{\sqrt{4x^2-9}}\,dx$

$\sqrt{4x^2-9} = \sqrt{4(x^2-\frac{9}{4})}$

$= 2\sqrt{x^2-(\frac{3}{2})^2}$

let $x = \frac{3}{2}\cosh u$

$\dfrac{dx}{du} = \dfrac{3}{2}\sinh u$

$\sqrt{x^2-(\frac{3}{2})^2} = \sqrt{(\frac{3}{2}\cosh u)^2 - (\frac{3}{2})^2}$

$= \frac{3}{2}\sqrt{\cosh^2 u - 1}$

$= \frac{3}{2}\sinh u$

$= \dfrac{1}{2}\displaystyle\int \frac{1}{\sqrt{x^2-(\frac{3}{2})^2}}\,dx$

$= \dfrac{1}{2}\displaystyle\int \frac{2}{3\sinh u} \times \frac{3\sinh u\,du}{2}$

$= \dfrac{1}{2}\displaystyle\int du$

$= \frac{1}{2}u$

$u = \operatorname{arcosh}\frac{2x}{3}$

$= \dfrac{1}{2}\left[\operatorname{arcosh}\dfrac{2x}{3}\right]_{2.5}^{3.9}$

$= \dfrac{1}{2}\left[\operatorname{arcosh}\left(\dfrac{2\times3.9}{3}\right) - \operatorname{arcosh}\left(\dfrac{2\times2.5}{3}\right)\right]$

$= \dfrac{1}{2}\left\{\ln\{2.6 + \sqrt{2.6^2-1}\} - \ln\{\tfrac{5}{3} + \sqrt{(\tfrac{5}{3})^2-1}\}\right\}$

$= \frac{1}{2}\ln 5 - \frac{1}{2}\ln 3$

$= \frac{1}{2}\ln\left(\frac{5}{3}\right)$

(iii) $y = \dfrac{\cosh x}{2+\sinh x}$ gradient $= \dfrac{1}{9}$

quotient rule $\dfrac{dy}{dx} = \dfrac{(2+\sinh x)(\sinh x) - (\cosh x)(\cosh x)}{(2+\sinh x)^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{2\sinh x + (\sinh^2 x - \cosh^2 x)}{(2+\sinh x)^2} = \dfrac{2\sinh x - 1}{(2+\sinh x)^2} = \dfrac{1}{9}$

$\therefore 9(2\sinh x - 1) = (2+\sinh x)^2$

$\therefore 18\sinh x - 9 = 4 + \sinh^2 x + 4\sinh x$

$\therefore \sinh^2 x - 14\sinh x + 13 = 0$

$(\sinh x - 1)(\sinh x - 13) = 0$

$\therefore \sinh x = 1$ or $\sinh x = 13$

If $\sinh x = 1$ then $\cosh(x) = \sqrt{2}$

$\therefore y = \dfrac{\sqrt{2}}{2+1} = \dfrac{\sqrt{2}}{3} \Rightarrow \left(\sinh^{-1}(1), \dfrac{\sqrt{2}}{3}\right)$

Coordinate is $\left(\ln(1+\sqrt{2}), \dfrac{\sqrt{2}}{3}\right)$ $\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$

Second point is where $\sinh x = 13$

$\therefore x = \sinh^{-1}(13) = \ln\left\{13 + \sqrt{13^2+1}\right\} = \ln\left\{13+\sqrt{170}\right\}$

$y = \dfrac{\cosh\left(\sinh^{-1}(13)\right)}{2+\sinh x}$ $\cosh^2 x = 1 + \sinh^2 x = 1 + 13^2 = 170$

$\therefore y = \dfrac{\sqrt{170}}{2+13} = \dfrac{\sqrt{170}}{15}$

Coordinate is $\left(\ln(13+\sqrt{170}), \dfrac{\sqrt{170}}{15}\right)$

FP2 June 2007.

4 (a) $\displaystyle\int_0^1 \frac{1}{\sqrt{9x^2+16}}\,dx = \int_0^1 \frac{1}{3\sqrt{x^2+\left(\frac{4}{3}\right)^2}}\,dx$

$= \frac{1}{3}\left[\operatorname{arsinh}\left(\frac{x}{4/3}\right)\right]_0^1$

$= \frac{1}{3}\left[\operatorname{arsinh}\left(\frac{3x}{4}\right)\right]_0^1$

$= \frac{1}{3}\left(\operatorname{arsinh}\left(\frac{3}{4}\right) - \operatorname{arsinh}(0)\right)$

$= \frac{1}{3} \ln\left\{\frac{3}{4} + \sqrt{1^2+\frac{9}{16}}\right\}$ $\qquad \frac{3}{4}+\frac{5}{4} = 2$

$= \frac{1}{3}\ln 2.$

(b) (i) Prove $\sinh 2x = 2\sinh x\cosh x$

$2\sinh x\cosh x = 2\left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}\right)$

$= \frac{1}{2}\left(e^x-e^{-x}\right)\left(e^x+e^{-x}\right)$

$= \frac{1}{2}\left(e^{2x}-e^{-2x}\right)$

$= \sinh(2x)$

(ii) $y = 20\cosh x - 3\cosh 2x$ $\qquad$ stationary points $\frac{dy}{dx}=0$

$\frac{dy}{dx} = 20\sinh x - 6\sinh 2x = 0$

$\therefore 20\sinh x - 6(2\sinh x\cosh x) = 0$

$\therefore 20\sinh x - 12\sinh x\cosh x = 0$

$\therefore 4\sinh x(5-3\cosh x) = 0$

$\therefore \sinh x = 0$ or $\cosh x = \frac{5}{3}$

$\therefore x = 0$ or $x = \ln\left\{\frac{5}{3}+\sqrt{\left(\frac{5}{3}\right)^2-1}\right\} = \ln 3$

when $x=0$ $\quad y = 20\times1 - 3\times1 = 17 \Rightarrow (0, 17)$

when $x=\ln 3$ $\quad y = 20\left(\frac{e^{\ln3}+e^{-\ln3}}{2}\right) - 3\left(\frac{e^{2\ln3}+e^{-2\ln3}}{2}\right)$

$\therefore y = 10\left(3+\frac{1}{3}\right) - \frac{3}{2}\left(9+\frac{1}{9}\right) = \frac{59}{3}$

$\Rightarrow \left(\ln 3, \frac{59}{3}\right)$

(iii) $\int_{-\ln 3}^{\ln 3} (20\cosh x - 3\cosh 2x)\,dx = 40$

$$\int_{-\ln 3}^{\ln 3} = 2\int_{0}^{\ln 3}$$

$$\int_{-\ln 3}^{\ln 3} \left(20\cosh x - 3\cosh 2x\right)\,dx$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$= \left[\, 20\sinh x - \frac{3}{2}\sinh 2x \right]_{-\ln 3}^{\ln 3}$$

$$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$$

$$= \left[20\left(\frac{e^{\ln 3} - e^{-\ln 3}}{2}\right) - \frac{3}{2}\left(\frac{e^{2\ln 3} - e^{-2\ln 3}}{2}\right) \right] \times 2$$

$$= \left[10\left(3 - \frac{1}{3}\right) - \frac{3}{4}\left(9 - \frac{1}{9}\right)\right] \times 2$$

$$= 20 \times 2$$

$$= 40.$$

FP2 January 2008

4 (i) $\cosh x = k \qquad k \geq 1$

$$\cosh x = \frac{e^x + e^{-x}}{2} = k \quad \Rightarrow \quad e^x + e^{-x} = 2k$$

$$\therefore e^{2x} + 1 = 2ke^x$$

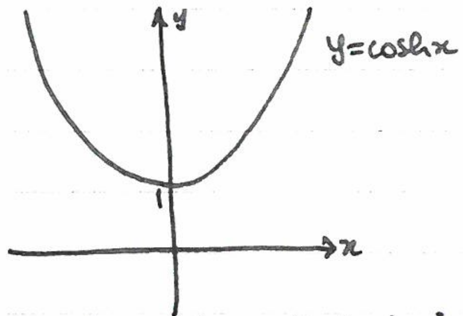
$$\therefore e^{2x} - 2ke^x + 1 = 0$$

$$\therefore e^x = \frac{2k \pm \sqrt{(2k)^2 - 4}}{2}$$

$$\therefore e^x = k \pm \tfrac{1}{2}\sqrt{4k^2 - 4} = k \pm \sqrt{k^2 - 1}$$

$$\therefore x = \ln\left\{k \pm \sqrt{k^2-1}\right\}$$

$$\therefore x = \ln\left\{k + \sqrt{k^2-1}\right\} \quad \text{or} \quad x = \ln\left\{k - \sqrt{k^2-1}\right\}$$



since $f(x) = f(-x)$ for $\cosh x$

if $x = \ln\left\{k + \sqrt{k^2-1}\right\}$

then $x = -\ln\left\{k + \sqrt{k^2-1}\right\}$

$\therefore x = \pm \ln\left\{k + \sqrt{k^2-1}\right\}$

even function - reflected in $x=0$

(ii)
$$\int_1^2 \frac{1}{\sqrt{4x^2-1}}\,dx = \int_1^2 \frac{1}{2\sqrt{x^2 - (\frac{1}{2})^2}}\,dx$$

$$= \frac{1}{2}\left[\operatorname{arcosh}\left(\frac{x}{1/2}\right)\right]_1^2$$

$$= \frac{1}{2}\left(\operatorname{arcosh}(4) - \operatorname{arcosh}(2)\right)$$

$$= \frac{1}{2}\left(\ln(4+\sqrt{15}) - \ln(2+\sqrt{3})\right)$$

(iii)
$$6\sinh x - \sinh 2x = 0$$

$$\therefore 6\sinh x - 2\sinh x \cosh x = 0$$

$$\therefore 2\sinh x\,(3 - \cosh x) = 0$$

$$\therefore \sinh x = 0 \quad \text{or} \quad \cosh x = 3$$

$$\therefore x = 0 \quad \text{or} \quad x = \cosh^{-1}(3) = \ln\left\{3 \pm \sqrt{8}\right\}$$

(iv) $y = 6\sinh x - \sinh 2x$ gradient $= 5$, no points exist.

$\frac{dy}{dx} = 6\cosh x - 2\cosh 2x = 5$

$\cosh 2x = 2\cosh^2 x - 1$

$\therefore 6\cosh x - 2(2\cosh^2 x - 1) = 5$

$\therefore 4\cosh^2 x - 6\cosh x + 3 = 0$

since $b^2 - 4ac \Rightarrow 36 - 4.3.4 = 36 - 48 < 0$

$\therefore$ no solution's for gradient $= 5$.

FP2 June 2008.

4 (i) Prove $\cosh^2 x - \sinh^2 x = 1$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\qquad \sinh x = \dfrac{e^x - e^{-x}}{2}$

$\therefore \cosh^2 x = \frac{1}{4}(e^x + e^{-x})^2 \qquad \sinh^2 x = \frac{1}{4}(e^x - e^{-x})^2$

$= \frac{1}{4}(e^{2x} + e^{-2x} + 2) \qquad = \frac{1}{4}(e^{2x} + e^{-2x} - 2)$

$\therefore \cosh^2 x - \sinh^2 x = \frac{1}{4}(e^{2x} + e^{-2x} + 2) - \frac{1}{4}(e^{2x} + e^{-2x} - 2)$

$= \frac{1}{4}(2 - (-2))$

$= \frac{1}{4} \times 4$

$= 1$

$\therefore \cosh^2 x - \sinh^2 x = 1$

(ii) $4\cosh^2 x + 9\sinh x = 13$

$\therefore 4(1 + \sinh^2 x) + 9\sinh x = 13$

$\therefore 4\sinh^2 x + 9\sinh x - 9 = 0$

$(4\sinh x - 3)(\sinh x + 3) = 0$

$\therefore \sinh x = \frac{3}{4}$ or $\sinh x = -3$

If $\sinh x = \frac{3}{4}$ then $x = \ln\left\{\frac{3}{4} + \sqrt{(\frac{3}{4})^2 + 1}\right\} = \ln 2$

If $\sinh x = -3$ then $x = \ln\left\{-3 + \sqrt{(-3)^2 + 1}\right\} = \ln(-3 + \sqrt{10})$

(iii) $y = 4\cosh^2 x + 9\sinh x$

$\dfrac{dy}{dx} = 8\cosh x \sinh x + 9\cosh x.$ $\qquad$ If $\dfrac{dy}{dx} = 0$ at st. pt.

$\therefore 8\cosh x \sinh x + 9\cosh x = 0$

$\cosh x(8\sinh x + 9) = 0$

$\therefore \cosh x = 0$ or $\sinh x = -\frac{9}{8}$ for $\dfrac{dy}{dx} = 0.$

Since $\cosh x \neq 0$ then $x = \operatorname{arsinh}(-\frac{9}{8})$ is only solution

So $y = 4(1 + \sinh^2 x) + 9\sinh x$

$\therefore y = 4\left(1 + \frac{81}{64}\right) + 9\left(-\frac{9}{8}\right) = -\frac{17}{16}.$

(iv) $\int_0^{\ln 2} (4\cosh^2 x + 9\sinh x)\,dx = 2\ln 2 + \frac{33}{8}$

$$\int_0^{\ln 2} (4\cosh^2 x + 9\sinh x)\,dx$$

$\cosh 2x = 2\cosh^2 x - 1$

$\cosh^2 x = \frac{\cosh 2x + 1}{2}$

$$= \int_0^{\ln 2} (2\cosh 2x + 2 + 9\sinh x)\,dx$$

$$= \Big[\sinh 2x + 2x + 9\cosh x\Big]_0^{\ln 2}$$

$$= \frac{1}{2}\left(e^{2\ln 2} - e^{-2\ln 2}\right) + 2\ln 2 + \frac{9}{2}\left(e^{\ln 2} + e^{-\ln 2}\right) - (0 + 0 + 9)$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right) + 2\ln 2 + \frac{9}{2}\left(2 + \frac{1}{2}\right) - 9$$

$$= 2\ln 2 + \frac{33}{8}$$

FP2 January 2009

4(a)(i) $\cosh x = \frac{1}{2}(e^{x}+e^{-x})$ $\quad \therefore \cosh^2 x = \frac{1}{4}(e^{x}+e^{-x})^2$

$\sinh x = \frac{1}{2}(e^{x}-e^{-x})$ $\quad \therefore \sinh^2 x = \frac{1}{4}(e^{x}-e^{-x})^2$

$$\therefore \cosh^2 x - \sinh^2 x = \frac{1}{4}(e^{2x}+e^{-2x}+2) - \frac{1}{4}(e^{2x}+e^{-2x}-2)$$

$$= \frac{1}{4}(e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2)$$

$$= \frac{4}{4}$$

$\therefore \cosh^2 x - \sinh^2 x = 1.$

(ii) $\sinh x = \tan y$ for $-\pi/2 < y < \frac{\pi}{2}$

(A) show $\tanh x = \sin y$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\tan y}{\cosh x}$$

but $\cosh^2 x = 1 + \sinh^2 x$

$\therefore \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\tan^2 y} = \sqrt{\sec^2 y}$

$\therefore \cosh x = \sec y$

$$\therefore \tanh x = \frac{\tan y}{\cosh x} = \frac{\sin y}{\cos y \,.\, \sec y} = \frac{\sin y}{\cos y} \,.\, \cos y$$

$\therefore \tanh x = \sin y$

(B) Show $x = \ln(\tan y + \sec y)$

$\sinh x = \tan y \Rightarrow x = \sinh^{-1}(\tan y)$

$$\therefore x = \ln\left\{\tan y + \sqrt{\tan^2 y + 1}\right\}$$

$$= \ln\left\{\tan y + \sqrt{\sec^2 y}\right\}$$

$$= \ln\left\{\tan y + \sec y\right\}$$

(b) (i) $y = \text{artanh}\, x$

$\therefore \tanh y = x$

$\therefore \dfrac{dx}{dy} = \text{sech}^2 y = 1 - \tanh^2 y = 1 - x^2$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{1-x^2}$

$\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\, dx = \Big[\text{artanh}\, x\Big]_{-1/2}^{1/2} = \text{artanh}\left(\tfrac{1}{2}\right) - \text{artanh}\left(-\tfrac{1}{2}\right)$

$= 2\,\text{artanh}\left(\tfrac{1}{2}\right)$

(ii) $\dfrac{1}{1-x^2} \equiv \dfrac{A}{(1+x)} + \dfrac{B}{(1-x)} \Rightarrow A(1-x) + B(1+x) = 1$

When $x = 1$ $\quad 2B = 1 \Rightarrow B = 1/2$

$x = -1$ $\quad 2A = 1 \Rightarrow A = 1/2$

$\therefore \dfrac{1}{(1-x^2)} = \dfrac{1}{2(1+x)} + \dfrac{1}{2(1-x)}$

$\displaystyle\int \frac{1}{(1-x^2)}\, dx = \int \frac{1}{2(1+x)}\, dx + \int \frac{1}{2(1-x)}\, dx$

$= \frac{1}{2}\ln|1+x| + \dfrac{\frac{1}{2}}{(-1)} \ln|1-x|$

$= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x|$

(iii) Show $\text{artanh}\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\ln 3$

From (i) $\displaystyle\int_{-1/2}^{1/2} \frac{1}{(1-x^2)}\, dx = 2\,\text{artanh}\left(\tfrac{1}{2}\right)$

From (ii) $\displaystyle\int_{-1/2}^{1/2} \frac{1}{(1-x^2)}\, dx = \frac{1}{2}\ln\left|\frac{1+\frac{1}{2}}{1-1/2}\right| - \frac{1}{2}\ln\left|\frac{1+(-\frac{1}{2})}{1-(-\frac{1}{2})}\right|$

$\therefore 2\,\text{artanh}\left(\tfrac{1}{2}\right) = \frac{1}{2}\ln|3| - \frac{1}{2}\ln\left|\frac{1}{3}\right| = \frac{1}{2}\left[\ln\left(\frac{3}{1/3}\right)\right]$

$\therefore 2\,\text{artanh}\left(\tfrac{1}{2}\right) = \frac{1}{2}\ln 9 = \ln 9^{1/2} = \ln 3$

$\therefore \text{artanh}\left(\tfrac{1}{2}\right) = \frac{1}{2}\ln 3.$

FP2 June 2009.

4 (i) Prove $\cosh 2x = 2\cosh^2 x - 1$

$$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$

$$= \frac{2}{4}\left(e^x + e^{-x}\right)^2 - 1$$

$$= \frac{1}{2}\left(e^{2x} + e^{-2x} + 2\right) - 1$$

$$= \frac{1}{2}\left(e^{2x} + e^{-2x}\right) + \frac{2}{2} - 1$$

$$= \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$$

$$= \cosh 2x.$$

(ii) prove $\operatorname{arsinh} y = \ln\left(y + \sqrt{y^2+1}\right)$

let $x = \operatorname{arsinh} y$

$\therefore\ y = \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

$\therefore\ 2y = e^x - e^{-x}$

$\therefore\ e^{2x} - 1 = 2ye^x$

$\therefore\ e^{2x} - 2ye^x - 1 = 0$

$\therefore\ e^x = \dfrac{2y \pm \sqrt{4y^2+4}}{2} = y \pm \sqrt{y^2+1}$

since $e^x > 0$ then $e^x = y + \sqrt{y^2+1}$ only.

$\therefore \ln e^x = \ln\left\{y + \sqrt{y^2+1}\right\}$

$\therefore\ x = \ln\left\{y + \sqrt{y^2+1}\right\}$

$\therefore\ \operatorname{arsinh} y = \ln\left\{y + \sqrt{y^2+1}\right\}$

(iii) $\displaystyle\int \sqrt{x^2+4}\, dx$

let $x = 2\sinh u$

$\dfrac{dx}{du} = 2\cosh u$

$$= \int \left(\sqrt{(2\sinh u)^2 + 4}\right)(2\cosh u)\, du$$

$4\sinh^2 u + 4$
$= 4\cosh^2 u$

$$= \int (2\cosh u)(2\cosh u)\, du$$

$2\cosh^2 u - 1 = \cosh 2u$

$4\cosh^2 u = 2\cosh 2u + 2$

$$= \int (2\cosh 2u + 2)\, du$$

$\therefore \int \sqrt{x^2+4}\, dx = \left[\sinh 2u + 2u \right] + c$

since
$x = 2\sinh u$
$u = \operatorname{arsinh}\left(\frac{x}{2}\right)$
$\sinh u = \frac{x}{2}$
$\cosh u = \sqrt{1+\sinh^2 u}$
$= \sqrt{1+\left(\frac{x}{2}\right)^2}$

$= 2\operatorname{arsinh}\left(\frac{x}{2}\right) + 2\sinh u \cosh u + c$

$= 2\operatorname{arsinh}\left(\frac{x}{2}\right) + 2\cdot\left(\frac{x}{2}\right)\left(1+\left(\frac{x}{2}\right)^2\right)^{1/2} + c$

$= 2\operatorname{arsinh}\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right)\left(\frac{4+x^2}{4}\right)^{1/2} + c$

$= 2\operatorname{arsinh}\left(\frac{x}{2}\right) + \frac{x}{2}\left(4+x^2\right)^{1/2} + c.$

(iv) $t^2 + 2t + 5 \equiv (t+1)^2 - 1 + 5$
$\equiv (t+1)^2 + 4$

$\int_{-1}^{1} \sqrt{t^2+2t+5}\, dt = \int_{-1}^{1} \sqrt{(t+1)^2+4}\, dt$

let $u = (t+1)$
$\frac{du}{dt} = 1$
$t=0 \quad u=1$
$t=1 \quad u=2$

$= \int_0^2 \sqrt{u^2+4}\, du$

$= \left[2\operatorname{arsinh}\left(\frac{u}{2}\right) + \frac{u}{2}\left(4+u^2\right)^{1/2} \right]_0^2$

$= 2\operatorname{arsinh}(1) + \sqrt{8} - 2\operatorname{arsinh}(0) - 0$

$= 2\ln\left\{1+\sqrt{1^2+1}\right\} + \sqrt{8}$

$= 2\ln(1+\sqrt{2}) + \sqrt{8}$
$= 2\ln(1+\sqrt{2}) + 2\sqrt{2}$
$= 2\left(\ln(1+\sqrt{2}) + \sqrt{2}\right).$

FP2 January 2010

4 (i) Prove $\cosh 2x = 1 + 2\sinh^2 x$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\quad$ $\sinh^2 x = \left(\dfrac{e^x - e^{-x}}{2}\right)^2$

$\therefore \sinh^2 x = \dfrac{1}{4}\left(e^{2x} + e^{-2x} + 2(e^x)(-e^{-x})\right)$

$= \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right)$

$= \dfrac{1}{4}\left(e^{2x} + e^{-2x}\right) - \dfrac{1}{2}$

$\therefore 1 + 2\sinh^2 x = 1 + 2\left\{\dfrac{1}{4}\left(e^{2x} + e^{-2x}\right) - \frac{1}{2}\right\}$

$= 1 + \frac{1}{2}\left(e^{2x} + e^{-2x}\right) - 1$

$= \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$

$= \cosh 2x$

If $\cosh 2x = 1 + 2\sinh^2 x$

differentiate both sides with respect to x

$\therefore 2\sinh 2x = 4\sinh x \cosh x$

$\therefore \sinh 2x = 2\sinh x \cosh x.$

(ii) $2\cosh 2x + 3\sinh x = 3$

$\therefore 2(1 + 2\sinh^2 x) + 3\sinh x = 3$

$\therefore 2 + 4\sinh^2 x + 3\sinh x = 3$

$\therefore 4\sinh^2 x + 3\sinh x - 1 = 0$

$\therefore (4\sinh x - 1)(\sinh x + 1) = 0$

$\therefore \sinh x = \frac{1}{4}$ or $\sinh x = -1$

$\therefore x = \ln\left\{\dfrac{1}{4} + \sqrt{\left(\dfrac{1}{4}\right)^2 + 1}\right\} = \ln\left\{\dfrac{1}{4} + \dfrac{\sqrt{17}}{4}\right\}$ $\quad$ arcsinh$(\frac{1}{4})$

or $x = \ln\left(-1 + \sqrt{(-1)^2 + 1}\right) = \ln\left\{-1 + \sqrt{2}\right\}$ $\quad$ arcsinh(-1)

(iii) $\cosh t = \frac{5}{4}$ $\quad \cosh t = \frac{e^{t}+e^{-t}}{2}$

$\therefore \frac{e^{t}+e^{-t}}{2} = \frac{5}{4} \Rightarrow e^{t}+e^{-t} = \frac{5}{2}$

$\therefore 2e^{t}+2e^{-t} = 5$

$\therefore 2e^{2t}+2 = 5e^{t}$

$\therefore 2e^{2t}-5e^{t}+2=0$

$\therefore (2e^{t}-1)(e^{t}-2) = 0$

$\therefore e^{t} = \frac{1}{2}$ or $e^{t} = 2$

$\therefore \ln e^{t} = \ln \frac{1}{2} \Rightarrow t = -\ln 2$

and $\ln e^{t} = \ln 2 \Rightarrow t = +\ln 2$

$\therefore t = \pm \ln 2$

From 1st principals

$$\int_4^5 \frac{1}{\sqrt{x^2-16}}\,dx$$

let $x = 4\cosh u$

$\frac{dx}{du} = 4\sinh u$

$x=4 \quad \cosh u = 1 \quad u=0$

$x=5 \quad \cosh u = \frac{5}{4}$

$$= \int \frac{1}{\sqrt{16\cosh^2 u-16}} \times 4\sinh u\,du$$

$$= \int \frac{1}{\sqrt{16(\cosh^2 u-1)}} \times 4\sinh u\,du$$

$$= \frac{1}{4}\int \frac{1}{\sqrt{\sinh^2 u}} \times 4\sinh u\,du$$

$$= \int du$$

$$= \left[u\right]_0^{\ln 2}$$

$$= (\ln 2) - 0$$

$$= \ln 2.$$

From Booklet

$$\int_4^5 \frac{1}{\sqrt{x^2-16}}\,dx = \int_4^5 \frac{1}{\sqrt{x^2-4^2}}\,dx = \left[\cosh^{-1}\left(\frac{x}{4}\right)\right]_4^5$$

$= \ln\{2\}$ from above since $\cosh^{-1}(1) = 0$.

FP2 June 2010

4 (i) Prove $\sinh 2x = 2\sinh x \cosh x$

$$2\sinh x\cosh x = 2\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left(e^{2x}+1-1-e^{-2x}\right)$$

$$= \frac{1}{2}\left(e^{2x}-e^{-2x}\right)$$

$$= \sinh 2x.$$

If $\sinh 2x = 2\sinh x \cosh x$

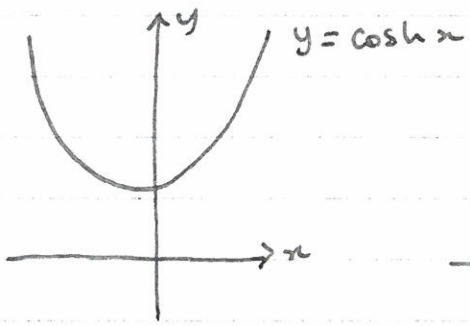
differentiate both sides with respect to x

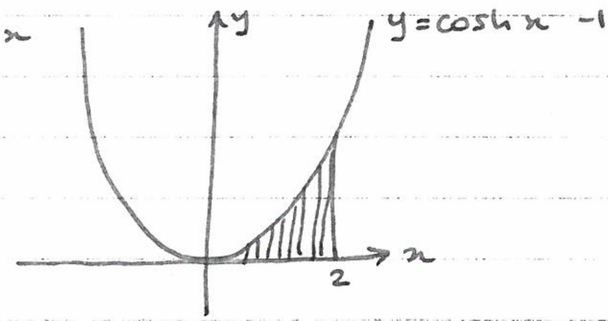
$$\therefore 2\cosh 2x = 2\cosh x\cosh x + 2\sinh x \sinh x$$

$$= 2\cosh^2 x + 2\sinh^2 x$$

$$\therefore \cosh 2x = \cosh^2 x + \sinh^2 x$$

(ii)





$$V = \pi\int y^2\,dx = \pi\int_0^2 (\cosh x - 1)^2\,dx$$

$$= \pi\int_0^2\left(\cosh^2 x + 1 - 2\cosh x\right)dx$$

$$= \pi\int_0^2\left(\frac{1}{2}\cosh 2x + \frac{3}{2} - 2\cosh x\right)dx$$

$$= \pi\left[\frac{1}{4}\sinh 2x + \frac{3}{2}x - 2\sinh x\right]_0^2$$

$$= \pi\left[\frac{1}{4}\sinh 4 + 3 - 2\sinh 2\right] - \pi\left[0\right]$$

$$= 2.568\pi$$

$$= 8.07 \text{ (3sf)} \quad 8.070 \text{ (3dpt)}.$$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$= \cosh^2 x + \cosh^2 x - 1$

$= 2\cosh^2 x - 1$

$\frac{1}{2}(\cosh 2x + 1) = \cosh^2 x$

(iii) $y = \cosh 2x + \sinh x$

$\frac{dy}{dx} = 2\sinh 2x + \cosh x$ For st pts $\frac{dy}{dx} = 0$

$\therefore 2\sinh 2x + \cosh x = 0$

But $\sinh 2x = 2\sinh x \cosh x$ (i)

$\therefore 2(2\sinh x \cosh x) + \cosh x = 0$

$\therefore 4\sinh x \cosh x + \cosh x = 0$

$\therefore \cosh x\,(4\sinh x + 1) = 0$

since $\cosh x \neq 0$

$\sinh x = -\frac{1}{4}$

$\therefore x = \operatorname{arcsinh}\left(\frac{-1}{4}\right)$

$\therefore x = \ln\left\{\left(-\frac{1}{4}\right) + \sqrt{\left(-\frac{1}{4}\right)^2 + 1}\right\} = \ln\left(-\frac{1}{4} + \frac{\sqrt{17}}{4}\right)$

FP2 January 2011.

4 (i) $\sinh t + 7\cosh t = 8$

$\therefore \left(\frac{e^t - e^{-t}}{2}\right) + 7\left(\frac{e^t + e^{-t}}{2}\right) = 8$

$\therefore 8e^t + 6e^{-t} = 16$

$\therefore 8e^{2t} + 6 - 16e^t = 0$

$\therefore 4e^{2t} - 8e^t + 3 = 0$

$(2e^t - 1)(2e^t - 3) = 0$

$\therefore e^t = \frac{1}{2}$ or $e^t = \frac{3}{2}$

$\therefore t = \ln\frac{1}{2} = -\ln 2$

or $t = \ln\frac{3}{2}$

(ii) $y = \cosh 2x + 7\sinh 2x$ coors where grad $= \frac{dy}{dx} = 16.$

$\frac{dy}{dx} = 2\sinh 2x + 14\cosh 2x$

If $2\sinh 2x + 14\cosh 2x = 16$

$\therefore \sinh 2x + 7\cosh 2x = 8$

from (i) $2x = \ln\left(\frac{1}{2}\right)$ $x = \frac{1}{2}\ln\left(\frac{1}{2}\right)$

or $2x = \ln\left(\frac{3}{2}\right)$ $x = \frac{1}{2}\ln\left(\frac{3}{2}\right)$

If $y = \frac{1}{2}(e^{2x} + e^{-2x}) + \frac{7}{2}(e^{2x} - e^{-2x})$

$x = \frac{1}{2}\ln\left(\frac{1}{2}\right) \Rightarrow y = \frac{1}{2}\left(e^{\ln\left(\frac{1}{2}\right)} + e^{-\ln\left(\frac{1}{2}\right)}\right) + \frac{7}{2}\left(e^{\ln\left(\frac{1}{2}\right)} - e^{-\ln\left(\frac{1}{2}\right)}\right)$

$\therefore y = 4\left(\frac{1}{2}\right) - 3(2) = -4$

$\left(\frac{1}{2}\ln\left(\frac{1}{2}\right), -4\right)$

$x = \frac{1}{2}\ln\left(\frac{3}{2}\right) \Rightarrow y = \frac{1}{2}\left(e^{\ln\left(\frac{3}{2}\right)} + e^{-\ln\left(\frac{3}{2}\right)}\right) + \frac{7}{2}\left(e^{\ln\left(\frac{3}{2}\right)} - e^{\ln\left(\frac{3}{2}\right)}\right)$

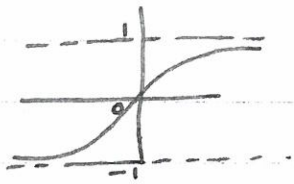
$\therefore y = 4\left(\frac{3}{2}\right) - 3\left(\frac{2}{3}\right) = 4$

$\left(\frac{1}{2}\ln\left(\frac{3}{2}\right), 4\right)$

If grad $= 0$ then $\sinh 2x + 7\cosh 2x = 0$

$\therefore \sinh 2x = -7\cosh 2x$

$\therefore \tanh 2x = -7$

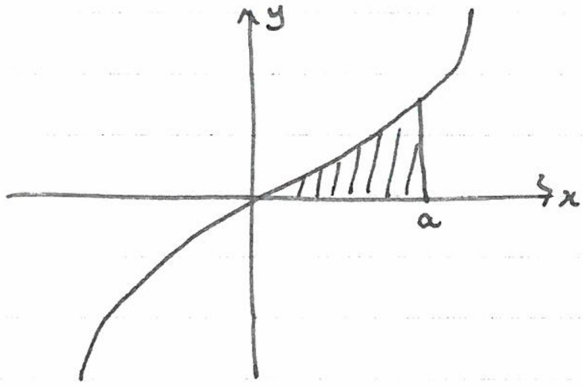


However, $-1 < \tan x < 1$

$-1 < \tan 2x < 1$

so no point on curve has a grad $= 0$

$\therefore$ no turning points



(iii) $\displaystyle\int_0^a (\cosh 2x + 7\sinh 2x)\, dx = \frac{1}{2}$

$\therefore \left[\frac{1}{2}\sinh 2x + \frac{7}{2}\cosh 2x\right]_0^a = \frac{1}{2}$

$\therefore \frac{1}{2}\sinh 2a + \frac{7}{2}\cosh 2a - \frac{1}{2}\sinh 0 - \frac{7}{2}\cosh 0 = \frac{1}{2}$

$\therefore \frac{1}{2}\sinh 2a + \frac{7}{2}\cosh 2a - \frac{7}{2} = \frac{1}{2}$

$\therefore \sinh 2a + 7\cosh 2a = 8$

$\therefore 2a = \ln\left(\frac{1}{2}\right)$ or $2a = \ln\left(\frac{3}{2}\right)$ from (i)

$\therefore a = \frac{1}{2}\ln\left(\frac{1}{2}\right)$ or $a = \frac{1}{2}\ln\left(\frac{3}{2}\right)$

Since $\frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln 2 < 0$, then $a = \frac{1}{2}\ln\left(\frac{3}{2}\right)$

FP2 June 2011.

4 (i) $\cosh y = x$

$\therefore \ x = \frac{1}{2}(e^{y} + e^{-y})$

$\therefore \ 2x = e^{y} + e^{-y}$

$\therefore \ e^{2y} + 1 - 2xe^{y} = 0$

$\therefore \ e^{2y} - 2xe^{y} + 1 = 0$

$\therefore \ e^{y} = \dfrac{+2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \frac{1}{2}\sqrt{4(x^2-1)} = x \pm \sqrt{x^2-1}$

$\therefore$ If $e^{y} = x \pm \sqrt{x^2-1}$

$\ln e^{y} = \ln(x + \sqrt{x^2-1})$ or $\ln e^{y} = \ln(x - \sqrt{x^2-1})$

$\therefore \ y = \ln(x + \sqrt{x^2-1})$ or $y = \ln(x - \sqrt{x^2-1})$

Taking the sum of the roots

$\ln(x + \sqrt{x^2-1}) + \ln(x - \sqrt{x^2-1})$

$= \ln(x + \sqrt{x^2-1})(x - \sqrt{x^2-1})$

$= \ln(x^2 - (x^2-1))$

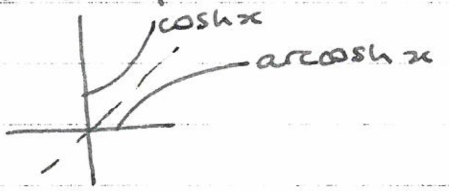
$= \ln(1)$

$= 0$

Therefore the roots are negatives of each other.

Since $\operatorname{arcosh} x > 0$

want only the positive root



$\therefore \ y = \pm \ln(x + \sqrt{x^2-1})$ principal value > 0

$\therefore \ y = \ln(x + \sqrt{x^2-1})$

(ii) $\displaystyle\int_{\frac{4}{5}}^{1} \frac{1}{\sqrt{25x^2-16}}\,dx = \int_{\frac{4}{5}}^{1} \frac{1}{\sqrt{25(x^2 - {}^{16}/_{25})}}\,dx = \int_{4/5}^{1} \frac{1}{5\sqrt{x^2 - (4/5)^2}}\,dx$

$= \frac{1}{5}\left[\ln\left\{\frac{5x}{4} + \sqrt{\left(\frac{5x}{4}\right)^2 - (1)^2}\right\}\right]_{4/5}^{1} \qquad = \left\{\operatorname{arcosh}\left(\frac{x}{4/5}\right)\right\}$

$= \frac{1}{5}\ln\left(\frac{5}{4} + \sqrt{\frac{25}{16} - 1}\right) - \frac{1}{5}\ln(1+0) \qquad = \left[\operatorname{arcosh}\left(\frac{5x}{4}\right)\right]_{4/5}^{1}$

(make $a=1$)

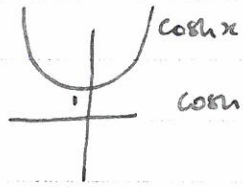
$$\therefore \int_{\frac{4}{5}}^{1} \frac{1}{\sqrt{25x^2-16}}\,dx = \frac{1}{5}\ln\left(\frac{5}{4}+\frac{3}{4}\right) - 0$$

$$= \frac{1}{5}\ln 2.$$

(iii) $5\cosh x - \cosh 2x = 3$ $\qquad \cosh 2x = 2\cosh^2 x - 1$

$\therefore 5\cosh x - 2\cosh^2 x + 1 = 3$

$\therefore 2\cosh^2 x - 5\cosh x + 2 = 0$



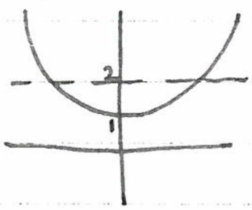
$\cosh x > 1$

$(2\cosh x - 1)(\cosh x - 2) = 0$

$\therefore \cosh x = \frac{1}{2}$ or $\cosh x = 2$

$\therefore x = \text{arcosh}(2) = \ln\left\{2 + \sqrt{2^2 - 1^2}\right\} = \ln(2+\sqrt{3})$

$\left[\text{Since } \cosh x \neq \frac{1}{2}\right]$



If $\cosh x = 2$ then $x = \pm\ln(2+\sqrt{3})$

or $x = \ln(2+\sqrt{3})$ and

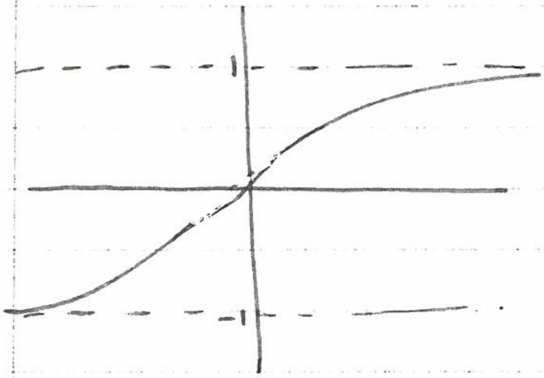
$x = \ln(2-\sqrt{3})$ *

* If in doubt : $x = -\ln(2+\sqrt{3}) = \ln(2+\sqrt{3})^{-1} = \ln\frac{1}{(2+\sqrt{3})}$

$$= \ln\left(\frac{(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}\right) = \ln\left(\frac{2-\sqrt{3}}{1}\right)$$

FP2 January 2012.

4 (i) $\tanh t = \dfrac{\sinh t}{\cosh t} = \dfrac{\frac{1}{2}(e^t - e^{-t})}{\frac{1}{2}(e^t + e^{-t})} = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$



(ii) Show $\operatorname{artanh} x = \frac{1}{2}\ln\left\{\dfrac{1+x}{1-x}\right\}$

If $x = \tanh y$ say, then $x = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$ from (i)

so $x e^y + x e^{-y} = e^y - e^{-y}$

$\therefore x e^{2y} + x - e^{2y} + 1 = 0$

$\therefore e^{2y}(x-1) + x + 1 = 0$

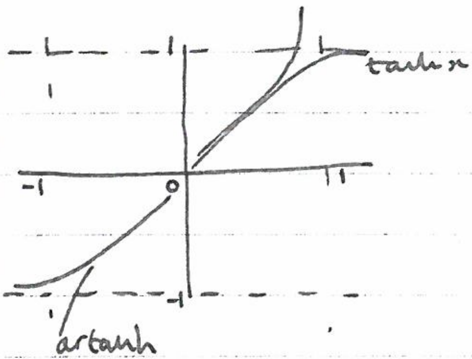
$\therefore e^{2y}(x-1) = -x-1$

$\therefore e^{2y} = -\dfrac{(x+1)}{(x-1)} = \dfrac{(x+1)}{(1-x)} = \dfrac{1+x}{1-x}$

$\therefore 2y = \ln\left[\dfrac{1+x}{1-x}\right]$

$\therefore y = \frac{1}{2}\ln\left[\dfrac{1+x}{1-x}\right]$

Since $y = \operatorname{artanh} x$ then $\operatorname{artanh} x = \frac{1}{2}\ln\left[\dfrac{1+x}{1-x}\right]$



so $-1 < x < 1$

(iii) $\tanh y = x$

differentiate with respect to x

$\therefore \operatorname{sech}^2 y \cdot \frac{dy}{dx} = 1$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \frac{1}{\cosh^2 x}$

$\therefore \operatorname{sech}^2 x = 1 - \tanh^2 x$

$\therefore \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1-\tanh^2 y}$

$\therefore \frac{dy}{dx} = \frac{1}{1-x^2}$ since $y = \operatorname{artanh} x$ this is the derivative of $\operatorname{artanh} x$

If $\operatorname{artanh} x = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right]$ and $\tanh y = x$

then $y = \frac{1}{2}\ln\left[\frac{1+x}{1-x}\right]$

differentiate with respect to x

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{\left[\frac{1+x}{1-x}\right]} \times \left\{\frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2}\right\}$$

$$= \frac{1}{2} \times \frac{(1-x)}{(1+x)} \times \frac{2}{(1-x)^2}$$

$$= \frac{1}{(1+x)(1-x)}$$

$= \frac{1}{1-x^2}$ which is also the derivative of $\operatorname{artanh} x$.

(iv) If $\operatorname{artanh} x = 1 \times \operatorname{artanh} x$ show

$$\int_0^{1/2} \operatorname{artanh} x = \frac{1}{4}\ln\left(\frac{27}{16}\right)$$

$\int u\,dv = uv - \int v\,du$

$u = \operatorname{artanh} x$

$du = \frac{1}{1-x^2}$

$dv = 1 \qquad v = x$

$$\therefore \int_0^{1/2} \operatorname{artanh} x = \left[x \operatorname{artanh} x\right]_0^{1/2} - \int_0^{1/2} \frac{x}{1-x^2}\,dx$$

$$= \left[x \operatorname{artanh} x\right]_0^{1/2} + \frac{1}{2}\int_0^{1/2} \frac{-2x}{1-x^2}\,dx$$

$$= \left[x \operatorname{artanh} x\right]_0^{1/2} + \frac{1}{2}\left[\ln(1-x^2)\right]_0^{1/2}$$

$$= \left\{\frac{1}{2}\operatorname{artan}\left(\frac{1}{2}\right) - 0\right\} + \frac{1}{2}\left\{\ln\left(\frac{3}{4}\right) - \ln(1)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{2}\ln\left(\frac{1+1/2}{1-1/2}\right)\right\} + \frac{1}{2}\ln\left(\frac{3}{4}\right) = \frac{1}{4}\ln\left(\frac{3/2}{1/2}\right) + \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

$$= \frac{1}{4}\ln(3) + \frac{1}{4}\ln\left(\frac{9}{16}\right) = \frac{1}{4}\ln\left(3 \times \frac{9}{16}\right) = \frac{1}{4}\ln\left(\frac{27}{16}\right)$$