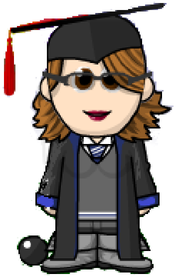
Differential Equations: Revision

MEI DE 4758



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Separation of Variables

A differential equation can be separated by its variables if it is of the form **(PRODUCT)**

$$\frac{dy}{dx} = f(x)g(y)$$

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

Example of separating variables

$$\frac{dy}{dx} = xy^2$$

Separate variables on either side of =

$$\int \frac{1}{y^2} dy = \int x dx$$

Integrate each side with respect to its variable

$$\int y^{-2} dy = \int x dx$$

$$-y^{-1} = \frac{1}{2}x^2 + c$$

Apply constants of integration to the right hand side only

$$-\frac{1}{y} = \frac{1}{2}x^2 + c$$

$$-\frac{1}{y} = \frac{x^2 + 2c}{2}$$

Rearrange the final equation to make y the subject since the DE was given as y differentiated with respect to x

$$\therefore y = -\frac{2}{x^2 + B}$$

Newton's Law of Cooling (use of modulus)

A drink sits in a room which has an ambient temperature of 20°C.

Its rate of temperature change is modelled by

$$\frac{dT}{dt} = -5(T-20)$$

Find the solutions to the differential equation when conditions are:

i. T=80 when t=0

ii. T=0 when t=0

$$\frac{dT}{dt} = -5(T-20)$$

$$\int \frac{dT}{(T-20)} = -5\int dt$$

$$\ln|T-20| = -5t + c$$

$$|T-20| = e^{-5t+c} = e^{-5t}e^{c} = Ae^{-5t}$$

$$\therefore |T-20| = Ae^{-5t}$$

i when t $= 0$, T $= 80$

$|T-20| > 0$, use $T-20$

$60 = Ae^{0} = A$

$\therefore T = 20 + 60e^{-5t}$

ii when t $= 0$, T $= 0$

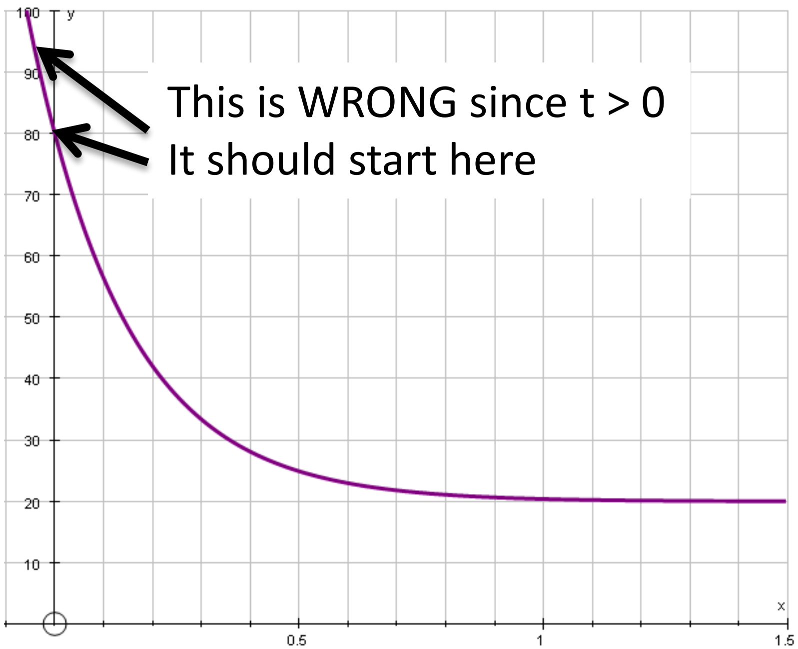
$|T-20| < 0$, use $20-T$

$20 = Ae^{0} = A$

$\therefore T = 20 - 20e^{-5t}$

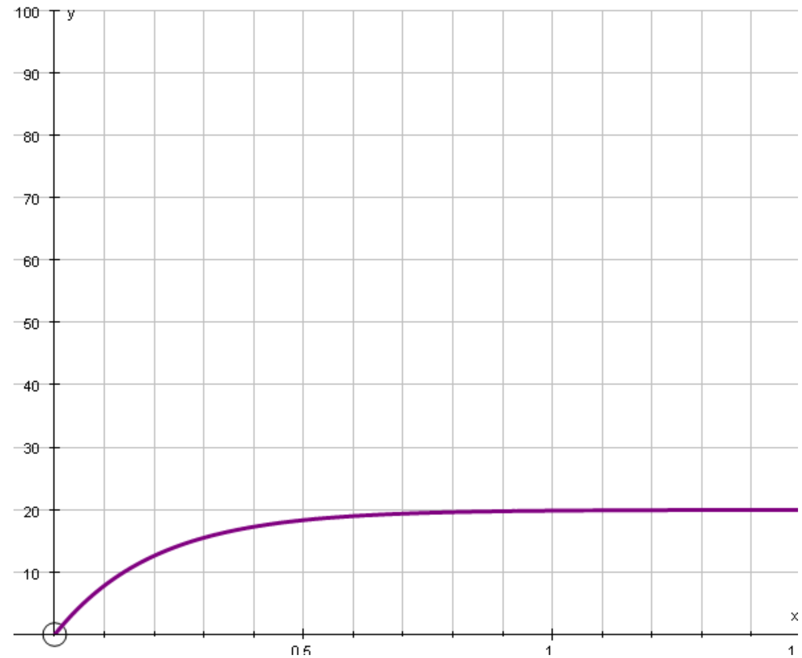
Newton’s Law of cooling example graphs

The drink cools down from 80° to 20°



$$T = 20 + 60e^{-5t}$$

The drink warms up from 0° to 20°



$$T = 20 - 20e^{-5t}$$

Integrating Factor Method

For DEs written as

$$\frac{dy}{dx} + P(x)y = Q(x)$$

P and Q are functions of x only (any degree)

$$\frac{dy}{dx} = x^2 - xy$$

$$\therefore \frac{dy}{dx} + xy = x^2 \qquad cf: \frac{dy}{dx} + P(x)y = Q(x)$$

$$So$$

$$P(x) = x$$

$$Q(x) = x^2$$

Integrating Factor Method

If the equation looks like

$$\frac{dy}{dx} + P(x)y = Q(x)$$

We want it in the form $$\frac{d}{dx}[R(x)y] = f(x)$$

Where R is a function in terms of x

And $f(x) = R(x)Q(x)$

The LHS of the equation will be a perfect derivative so there is only the RHS to integrate

R is the integrating factor where $$R = e^{\int P(x)dx}$$

Example using integrating factor R

$$\frac{dy}{dx}+\frac{2y}{x}=\frac{4}{x^2} \qquad cf: \frac{dy}{dx}+P(x)y=Q(x)$$

- Rearrange into linear form if necessary

$$R=e^{\int P dx}=e^{\int \frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$$

- Find the integrating factor R

$$x^2\frac{dy}{dx}+x^2\frac{2y}{x}=x^2\frac{4}{x^2}$$

- Multiply **BOTH SIDES** through the equation by R

$$\therefore \frac{d}{dx}(x^2y)=4$$

- The left hand side is now a perfect derivative

$$\therefore x^2y=\int 4dx=4x+c$$

- The right hand side is a function of x and can be integrated

$$y=\frac{4}{x}+\frac{c}{x^2}$$

- Rearrange to give y

Example using Integrating Factor

$$\cos x \frac{dy}{dx} - (\sin x)y = x^2$$

$$\Rightarrow \quad \frac{dy}{dx} - \left(\frac{\sin x}{\cos x}\right)y = \frac{x^2}{\cos x} \qquad R = e^{-\int \frac{\sin x}{\cos x}dx} = e^{\ln \cos x} = \cos x$$

$$\frac{d}{dx}(y\cos x) = \frac{x^2}{\cos x} \times \cos x$$

$$\therefore y\cos x = \int x^2 dx = \frac{x^3}{3} + c$$

$$y = \frac{x^3}{\cos x} + \frac{c}{\cos x} \qquad \text{(General Solution)}$$

General & Particular Solutions

$$x^2\frac{dy}{dx}+xy=\frac{2}{x} \quad when \quad y=1,\ \ x=2$$

$$\frac{dy}{dx}+\frac{xy}{x^2}=\frac{2}{x^3} \qquad \Rightarrow \quad R=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$$

$$\frac{d}{dx}(xy)=\frac{2}{x^3}\times x$$

$$xy=\int\frac{2}{x^2}dx=-\frac{2}{x}+c$$

$$\therefore\ y=-\frac{2}{x^2}+\frac{c}{x} \quad \text{(General Solution)}$$

$$y=1,\ \ x=2 \quad \Rightarrow \quad 1=-\frac{2}{3}+\frac{c}{2} \quad \Rightarrow \quad \therefore c=3$$

$$\therefore\ y=-\frac{2}{x^2}+\frac{3}{x} \quad \text{This is the particular solution}$$

Homogeneous Equations [...=0]

1st order linear

$$5\frac{dy}{dx}+y=0$$

2nd order linear

$$\frac{d^2y}{dx^2}+4y=0$$

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

Auxiliary Equations: single root

For a linear 1st order equation of the type

$$5\frac{dy}{dx} + y = 0$$

Assume $y = Ae^{\lambda x} \Rightarrow y' = A\lambda e^{\lambda x}$

Substitute into the DE

$5A\lambda e^{\lambda x} + Ae^{\lambda x} = 0$

$Ae^{\lambda x}(5\lambda + 1) = 0$

Since $e^{\lambda x} \neq 0$,

The Auxiliary Equation is $5\lambda + 1 = 0$

$\Rightarrow \lambda = -\frac{1}{5}$

The Complementary Function is $y = Ae^{-\frac{x}{5}}$

Compare with solving by:

$5\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = -\frac{y}{5}$

$\frac{dy}{y} = -\frac{dx}{5}$

$\int \frac{dy}{y} = \int -\frac{dx}{5}$

$\ln y = -\frac{1}{5}x + c$

$y = e^{-\frac{1}{5}x + c} = e^{-\frac{1}{5}x}.e^{c} = Ae^{-\frac{1}{5}x}$

Auxiliary Equations: 2 real roots

For a linear 2nd order equation of the type

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

$$use$$

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

Auxiliary equation $\lambda^2 - 5\lambda + 6 = 0$

$$\Rightarrow \lambda = 2, \lambda = 3$$

Complementary function $\underline{y = Ae^{2x} + Be^{3x}}$

Auxiliary Equations: pure imaginary roots

For a linear 2nd order equation of the type

$$\frac{d^2y}{dx^2} + 4y = 0$$

use

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

Auxiliary equation

$$\lambda^2 + 4 = 0$$

$$\Rightarrow \lambda = \pm 2j$$

Complementary function

$$y = A\sin 2x + B\cos 2x$$

Auxiliary Equations: general complex roots

For a linear 2nd order equation of the type

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

Auxiliary equation

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\Rightarrow \lambda = -1 \pm 2j$$

Complementary function

$$y = e^{-1x}(Ae^{2jx} + Be^{-2jx})$$

$$y = e^{-x}(A\cos 2x + B\sin 2x)$$

Auxiliary equation: repeated roots

For a linear 2[nd] order equation of the type

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Auxiliary equation

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

Complementary function

$$y = Ae^{\lambda x} + Bxe^{\lambda x}$$

$$y = Ae^{2x} + Bxe^{2x}$$

Complementary Functions

Roots for Aux Equation	Complementary Function
$\lambda^2 + a\lambda + b = 0$	
$\lambda = \lambda_1, \quad \lambda_2$	$y = Ae^{\lambda_1 x} + \mathrm{B}e^{\lambda_2 x}$
$\lambda = \alpha \pm j\beta$	$y = e^{\alpha x}(A\sin\beta x + B\cos\beta x)$
$\lambda = m$ (twice)	$y = Ae^{mx} + \mathrm{B}xe^{mx}$

Non-homogeneous Equations [...=f(x)]

Compare the solutions for the following

$$\frac{dy}{dx} + 2y = e^{3x}$$
$$\frac{dy}{dx} + 2y = 3x - 1$$

The auxiliary equation for the homogenous DE is the same in both cases and therefore both equations will have the same Complementary Function, only the integrated function on the RHS of the DE will be different

We choose a function similar to the RHS (with undefined constants) which we substitute into the DE to find the Particular Integral part of the general solution

Particular Integral for e^{3x}

Let $$\frac{dy}{dx}+2y=e^{3x}$$

You have to find the CF first, as the PI must not already be part of the solution – here we try a multiple of e^{3x}

Auxiliary Equation

$$LHS: \quad \lambda+2=0$$

$$\therefore \lambda=-2$$

Complementary Function

$$y=Ae^{-2x}$$

Particular Integral

$$RHS: \quad y=ce^{3x}$$

$$\therefore \frac{dy}{dx}=3ce^{3x}$$

Substitute into the DE

$$3ce^{3x}+2ce^{3x}=e^{3x}$$

$$\therefore c=\frac{1}{5}$$

General Solution $=CF + PI$

$$y=Ae^{-2x}+\frac{1}{5}e^{3x}$$

Particular Integral for 3x-1

The PI will be a linear function in x

Let $$\frac{dy}{dx}+2y=3x-1$$

$\text{Auxiliary Equation}$

$LHS: \quad \lambda+2=0$

$\therefore \lambda=-2$

$\text{Complement ary Function}$

$y=Ae^{-2x}$

$\text{Particular Integral}$

$RHS: \quad y=ax+b$

$\therefore \dfrac{dy}{dx}=a$

$a+2(ax+b)=3x-1$

$\Rightarrow 2a=3 \qquad \Rightarrow a=\dfrac{3}{2}$

$\Rightarrow a+2b=-1 \qquad \Rightarrow b=-\dfrac{5}{4}$

General

Solution $CF\ \&\ PI$

$$y=Ae^{-2x}+\frac{3}{2}x-\frac{5}{4}$$

Particular Integral Types

Function f(x)	Particular Integral Type
constant	k
linear	$ax+b$
quadratic	ax^2+bx+c
sin or cos	$a\sin wx+b\cos wx$
exponential	ce^{px}
if the trial function gives $=0$ then use a product with x	cxe^{px}
if a product of x already exists, then try one with x^2	cx^2e^{px}

PI does not work [substitution=0]

Explain why the particular integral for $\dfrac{dy}{dt} + 2y = e^{-2t}$ cannot take the form $y = ae^{-2t}$

$\dot{y} + 2y = 0$

$LHS:$ Aux Equation

$\lambda + 2 = 0$

$\lambda = -2$

$CF: \ y = Ae^{\lambda t} = Ae^{-2t}$

$\dot{y} + 2y = e^{-2t}$

$RHS:$ Particular Integral

Try $y = ae^{-2t} \Rightarrow \dot{y} = -2ae^{-2t}$

$\therefore -2ae^{-2t} + 2ae^{-2t} = ae^{-2t}$

$\Rightarrow 0 = ae^{-2t}$

The Complementary Function already contains e^{-2t} so trying a Particular Integral of the same type gives 0 on substitution

Try $y = ate^{-2t} \Rightarrow \dot{y} = -2ate^{-2t} + ae^{-2t}$

$\therefore -2ate^{-2t} + ae^{-2t} + 2ae^{-2t} = ate^{-2t}$

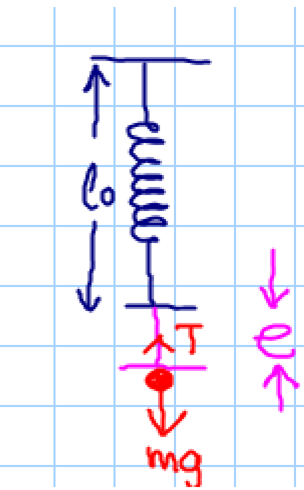
$\Rightarrow a = 1$

$PI: \ y = ate^{-2t} = te^{-2t}$

$GS: \ y = Ae^{-2t} + te^{-2t}$

Simple Harmonic Motion

Oscillating systems without damping: vertical spring



At rest:

$$T = ke$$

$$T = mg$$

$$\therefore e = \frac{mg}{k}$$

In motion: F=ma

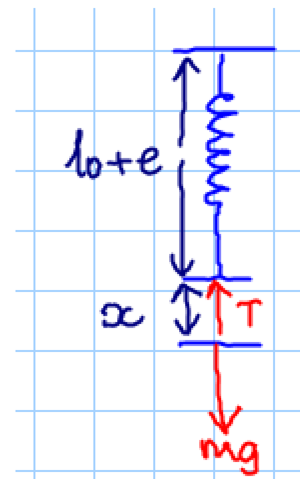
$$T = k(e + x) = mg + kx$$

$$mg - T = ma = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} = mg - ke - kx$$

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

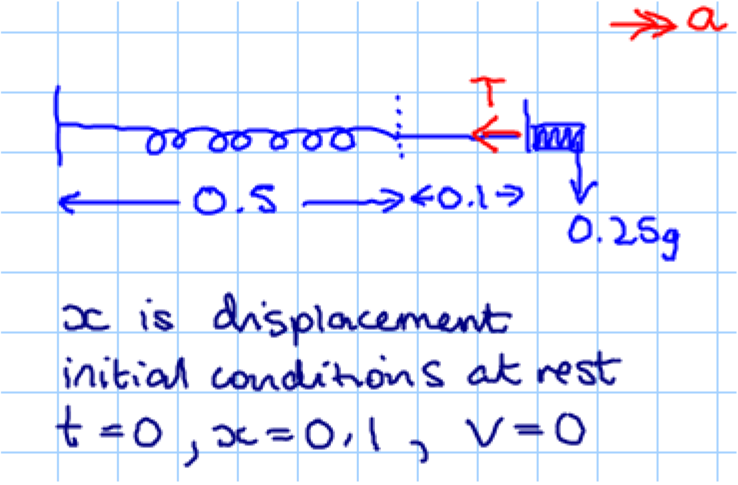


Simple Harmonic Motion

Oscillating systems without damping: horizontal spring

At rest:

$$T = kx$$



In motion: F=ma

$$-T = m\frac{d^2x}{dt^2}$$

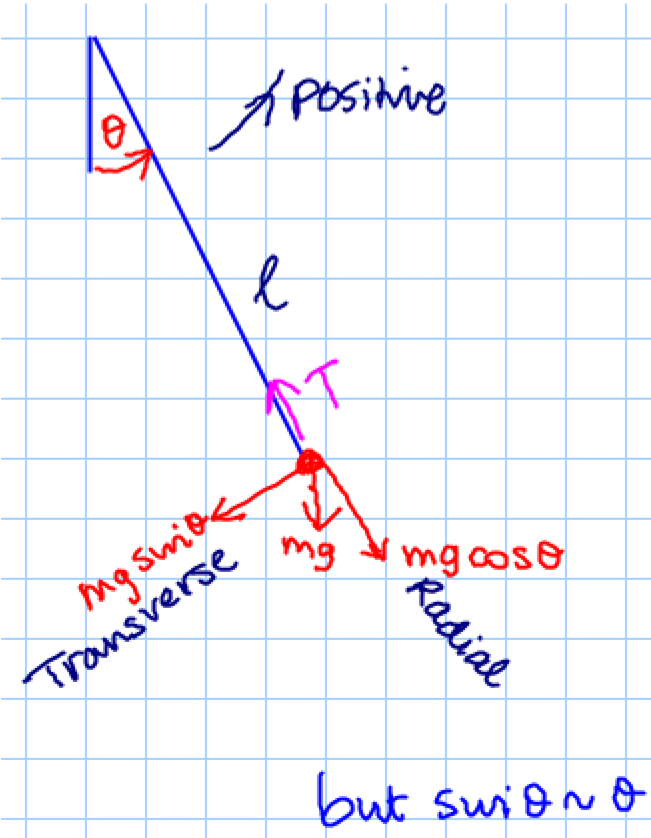
$$-kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Simple Harmonic Motion

Oscillating systems without damping: pendulum



In motion: F=ma (transverse)

$$-mg\sin\theta = ml\frac{d^2\theta}{dt^2}$$

$$\theta \to 0, \sin\theta \to \theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

Velocity and acceleration are dependent on length

SHM – General Case

For the general case without damping

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad \text{where} \quad \omega^2 = -\frac{k}{m}, or \quad \omega^2 = -\frac{g}{l}$$

$$\text{Aux Equation}: \lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm \omega j$$

$$\therefore x = A\sin\omega t + B\cos\omega t$$

$$\text{Let} \quad R\sin(\omega t + \varepsilon) \equiv x$$

$$\Rightarrow R\sin\omega t\cos\varepsilon + R\cos\omega t\sin\varepsilon \equiv A\sin\omega t + B\cos\omega t$$

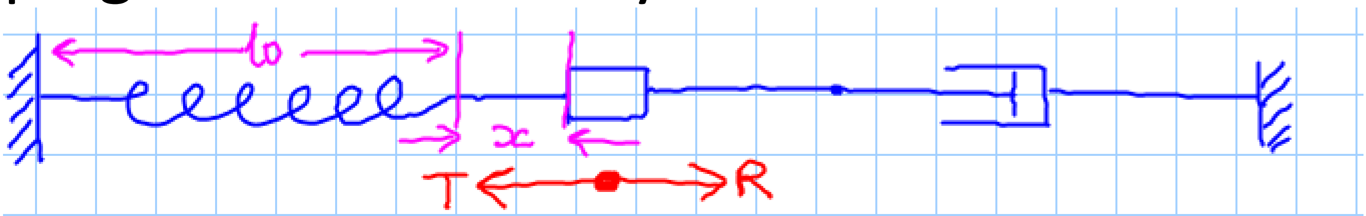
$$\therefore R = \sqrt{A^2 + B^2} = \text{amplitude}$$

$$R\cos\varepsilon = A, R\sin\varepsilon = B \Rightarrow \tan\varepsilon = \frac{B}{A}$$

$$\text{Phase} = \varepsilon \qquad \text{translation is } \frac{\varepsilon}{\omega} \qquad \text{Period} = \frac{2\pi}{\omega} \qquad \text{Frequency} = \frac{\omega}{2\pi}$$

SHM with damping

Damping introduces a way to reduce oscillations



At rest: tension→ $T = kx$

Damping force→ $R = -r\frac{dx}{dt}$

N2LM "F=ma" $R - T = ma$

$$-r\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

General Damped Equations

Consider the discriminant of the auxiliary equation for

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \omega^2 y = 0 \quad \text{where} \quad \alpha = \frac{\text{r}}{\text{m}} \text{ and } \omega^2 = \frac{k}{m}$$

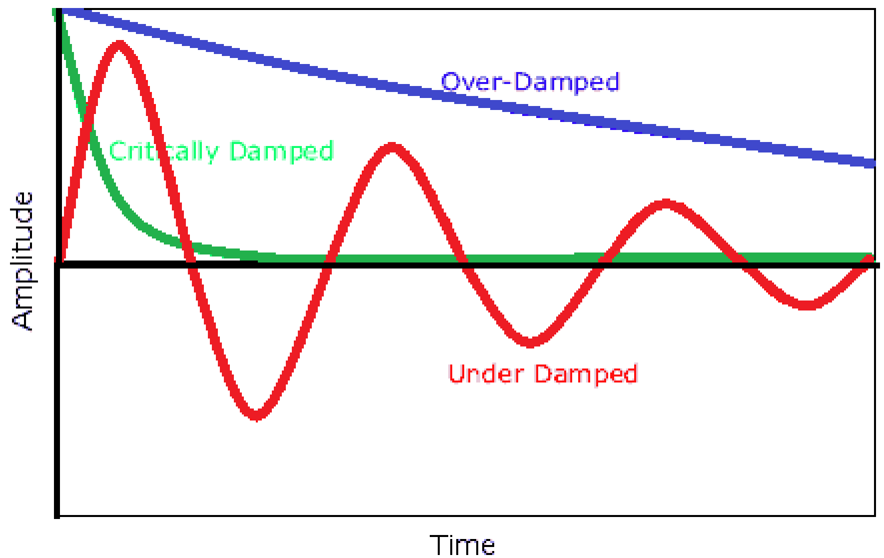
$$\text{Aux Eqn}: \quad \lambda^2 + \alpha\lambda + \omega^2 = 0$$

$$\text{Discriminant}: \quad \alpha^2 - 4\omega^2$$

$$\alpha^2 - 4\omega^2 > 0 \Rightarrow overdamping$$

$$\alpha^2 - 4\omega^2 < 0 \Rightarrow underdamping$$

$$\alpha^2 - 4\omega^2 = 0 \Rightarrow critical\ damping$$



Critically Damped Equations

An object of mass 0.25kg has undamped motion described by $\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad (k = 20,\ \text{and}\ \ \omega^2 = 80)$

What is the value of the damping constant if the system is to be critically damped?

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \omega^2 y = 0 \quad \text{where } \alpha = \frac{r}{m} \text{ (r = damping) and } \omega^2 = 80$$

$$\text{Aux Eqn: } \lambda^2 + \alpha\lambda + 80 = 0$$

$$\text{Discriminant: } \alpha^2 - 4\times 80 = 0 \Rightarrow \textit{critical damping}$$

$$\therefore \alpha^2 = 320 \Rightarrow r^2 = 320\times 0.25^2$$

$$\therefore r = \sqrt{20}$$

Critically Damped Equations

What happens to the same system if the object has its mass i) Increased to 0.3kg ii) Decreased to 0.2kg given that all other constants remain unchanged

$$\frac{d^2y}{dt^2}+\alpha\frac{dy}{dt}+\omega^2 y=0 \quad \text{where } \alpha=\frac{r}{m}\,(\text{r} = \text{damping}) \text{ and } \omega^2=\frac{20}{m}$$

$$\text{Aux Eqn: } \lambda^2+\alpha\lambda+\omega^2=0 \Rightarrow \lambda^2+\frac{r}{m}\lambda+\frac{20}{m}=0$$

$$\text{Discriminant: } \alpha^2-4\omega^2=\left(\frac{r}{m}\right)^2-4\left(\frac{20}{m}\right)=0 \Rightarrow critical\ damping$$

$$m=0.3 \Rightarrow \frac{20}{0.09}-\frac{80}{0.3}<0 \Rightarrow underdamped$$

$$m=0.2 \Rightarrow \frac{20}{0.04}-\frac{80}{0.2}>0 \Rightarrow overdamped$$

General SHM solutions

(Remember modelling assumptions for SHM)

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$\alpha = 0$	$x = A\sin\omega t + B\cos\omega t$ $x = a\sin(\omega t + \varepsilon)$	no damping

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega^2 x = 0$$

$\alpha > 2\omega$	$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$	over damped
$\alpha = 2\omega$	$x = (A + Bt)e^{-\frac{\alpha}{2}t}$	critically damped
$\alpha < 2\omega$	$x = ae^{-\frac{\alpha}{2}t}\sin(pt + \varepsilon)$ $p = \frac{1}{2}\sqrt{4\omega^2 - \alpha^2}$	under damped

Systems of DEs

Solve for the differential equations:

$$(1)\ \frac{dx}{dt} = a_1 x + b_1 y + c_1$$

$$(2)\ \frac{dy}{dt} = a_2 x + b_2 y + c_2$$

Normally we eliminate y and $\dot{y}$

$rearrange:(1)$ (3) $b_1 y = \dot{x} - a_1 x - c_1$

$dwrt:t$ (4) $b_1 \dot{y} = \ddot{x} - a_1 \dot{x}$

The independent variable is the one the others are differentiated with respect to

$substitute:(3)\ \&\ (4) \Rightarrow (2)$

$$\frac{1}{b_1}(\ddot{x} - a_1 \dot{x}) = a_2 x + \frac{b_2}{b_1}(\dot{x} - a_1 x - c_1) + c_2$$

$rearrange$

$$\frac{1}{b_1}\ddot{x} - \frac{1}{b_1}(a_1 - b_2)\dot{x} + \left(a_1 \frac{b_2}{b_1} - a_2\right)x = c_2$$

Now we can solve using Auxiliary equations and Particular Integral methods

System of DEs Example – General Solution

Solve for (1) $\frac{dx}{dt} = -x + y - 1$

(2) $\frac{dy}{dt} = -x - y + 3$

- Firstly, eliminate y
- Solve for x
- Then go back and solve for y

$$(1) \quad \dot{x} = -x + y - 1$$
$$\Rightarrow y = \dot{x} + x + 1 \quad (3)$$
$$\Rightarrow \dot{y} = \ddot{x} + \dot{x} \qquad (4)$$

$$(3)\ \&\ (4) \Rightarrow (2)$$
$$\ddot{x} + \dot{x} = -x - (\dot{x} + x + 1) + 3$$
$$\ddot{x} + 2\dot{x} + 2x = 2$$

$$\ddot{x} + 2\dot{x} + 2x = 2$$
$$AE: \lambda^2 + 2\lambda + 2 = 0$$
$$(\lambda + 1)^2 + 1 = 0$$
$$\Rightarrow \lambda = -1 \pm j$$
$$CF: x = e^{-t}(A\sin t + B\cos t)$$
$$PI: x = k$$
$$2k = 2 \Rightarrow k = 1$$

$$GS:$$
$$x = e^{-t}(A\sin t + B\cos t) + 1$$

System of DEs Example – General Solution

Solve for (1) $\frac{dx}{dt} = -x + y - 1$ (2) $\frac{dy}{dt} = -x - y + 3$

$$y = \dot{x} + x + 1 \quad (3)$$

$$x = e^{-t}(A\sin t + B\cos t) + 1$$

$$\dot{x} = -e^{-t}(A\sin t + B\cos t) + e^{-t}(A\cos t - B\sin t)$$

$$\therefore \dot{x} = 1 - x + e^{-t}(A\cos t - B\sin t)$$

$$\therefore y = [1 - x + e^{-t}(A\cos t - B\sin t)] + x + 1$$

$$\therefore y = e^{-t}(A\cos t - B\sin t) + 2$$

System of DEs Example – Particular Solution

Solve for (1) $\frac{dx}{dt} = -x + y - 1$ (2) $\frac{dy}{dt} = -x - y + 3$

And when t=0, x=0, y=3

- Make the substitution for x at the GS stage

$$x = e^{-t}(A\sin t + B\cos t) + 1$$
$$t = 0, x = 0$$
$$0 = B + 1 \Rightarrow B = -1$$
$$y = e^{-t}(A\cos t - B\sin t) + 2$$
$$t = 0, y = 3$$
$$3 = A + 2 \Rightarrow A = 1$$

$$x = e^{-t}(\sin t - \cos t) + 1$$
$$y = e^{-t}(\cos t + \sin t) + 2$$

System of DEs Example – Limits & Bounds

Find the limit of x and y when t is very large and positive

If $\frac{y}{x}=k$ when t is very large, find k

$$x = e^{-t}(\sin t - \cos t) + 1$$

$$t \to \infty, e^{-t} \to 0, x \to 1$$

$$y = e^{-t}(\cos t + \sin t) + 2$$

$$t \to \infty, e^{-t} \to 0, y \to 2$$

$$\frac{y}{x} \to \frac{2}{1} \Rightarrow k = 2$$

If a solution looks like

$$x = Ae^{-t} + Be^{3t} + 1$$

and it is said to be bounded for large positive values of t, then the coefficient of e^{3t} must be zero, B=0

Similarly if it is said to be bounded for large negative values of t, then e^{-t} must be zero, so A=0

Equilibrium Points

A solution curve plots the motion of the dependent variables from the initial point to some end point

There is a direction associated with solution curves, so they should have arrows on them

Solution curves can be drawn from tangent fields and a particular one highlighted from a family of curves

Equilibrium Points Case 1

For

(1) $\frac{dx}{dt} = 2x + 4y$

(2) $\frac{dy}{dt} = x - y$

$\therefore \frac{dy}{dx} = \frac{x - y}{2x + 4y}$

Initially

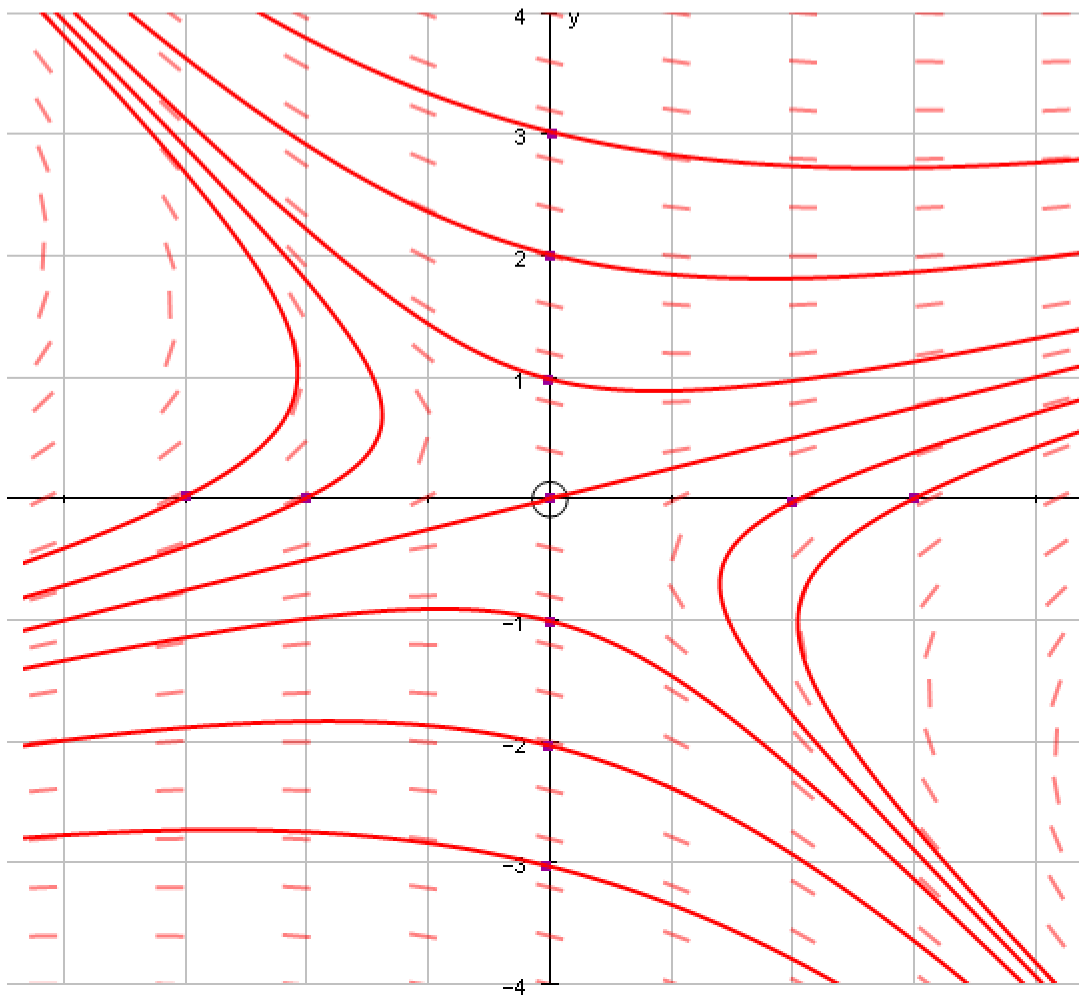
$t = 0$

$x = 2$

$y = -2$

$\frac{dx}{dt} = 0 \Rightarrow y = -x$

$\frac{dy}{dt} = 0 \Rightarrow y = 0.25x$



Equilibrium point is (0,0)

Equilibrium Points Case 2

For

$$(1)\quad \frac{dx}{dt} = -3y$$

$$(2)\quad \frac{dy}{dt} = 3x$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

Initially

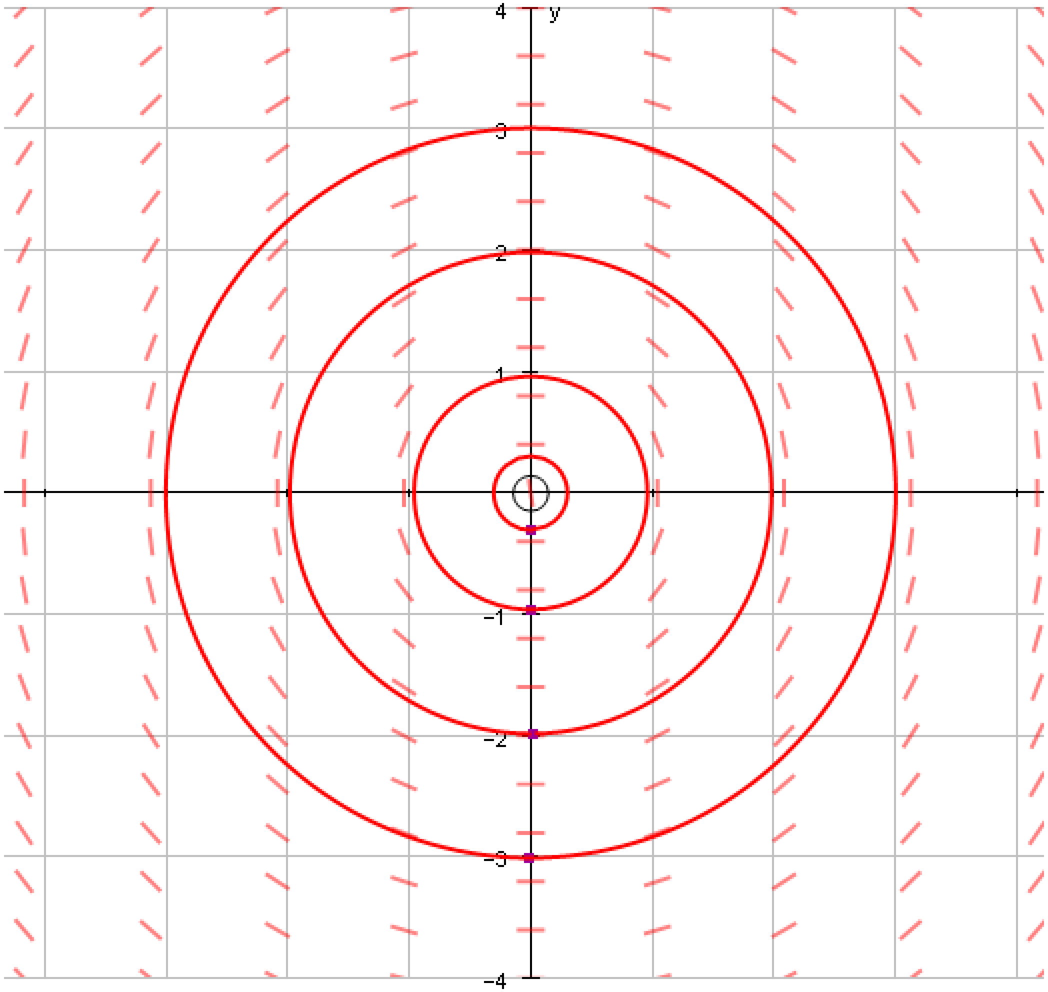
$t = 0$

$x = 3$

$y = 4$

$$\frac{dx}{dt} = 0 \Rightarrow x = 0$$

$$\frac{dy}{dt} = 0 \Rightarrow y = 0$$



Equilibrium point is (0,0)

Equilibrium Points Case 3

For

(1) $$\frac{dx}{dt} = -x + y - 1$$

(2) $$\frac{dy}{dt} = -x - y + 3$$

$$\therefore \frac{dy}{dx} = \frac{-x - y + 3}{-x + y - 1}$$

Initially

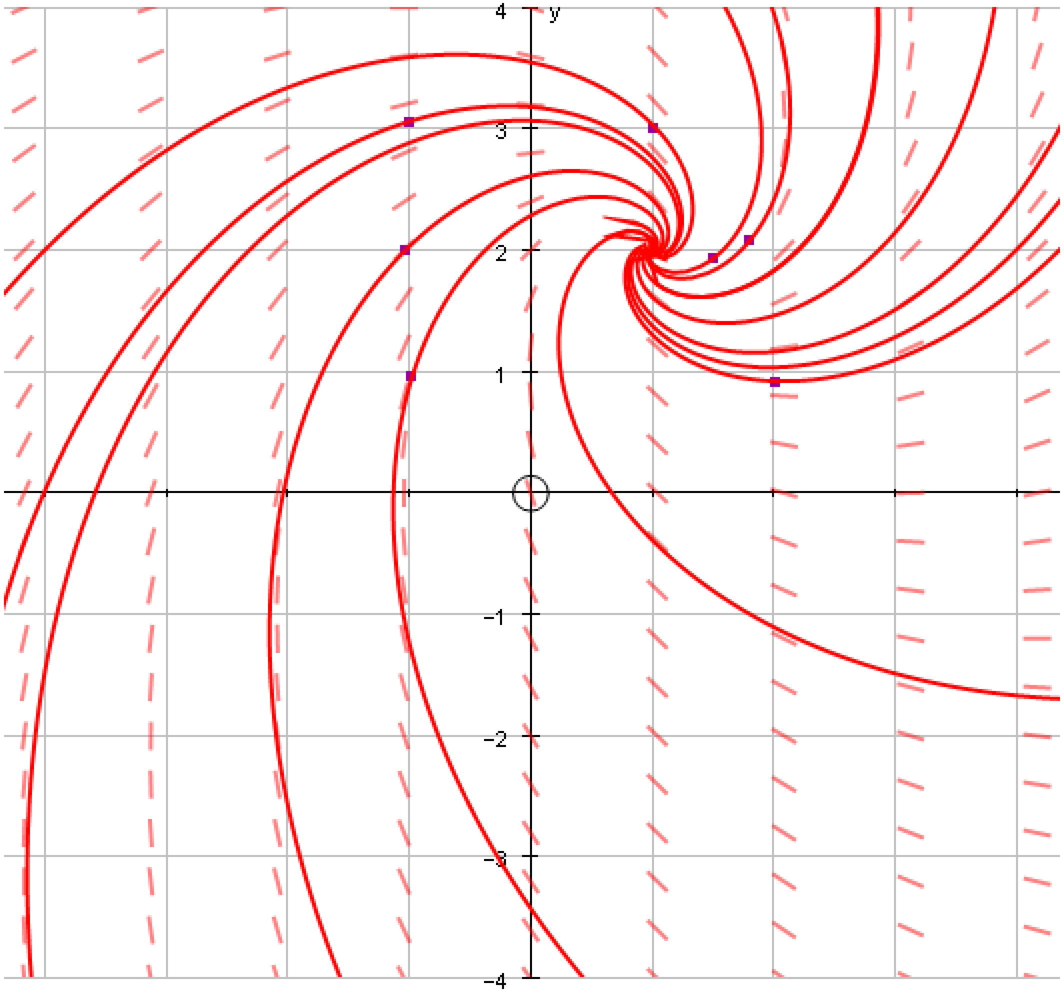
$t = 0$

$x = 0$

$y = 3$

$$\frac{dx}{dt} = 0 \Rightarrow y = x - 1$$

$$\frac{dy}{dt} = 0 \Rightarrow y = 3 - x$$



Equilibrium point is (1,2)

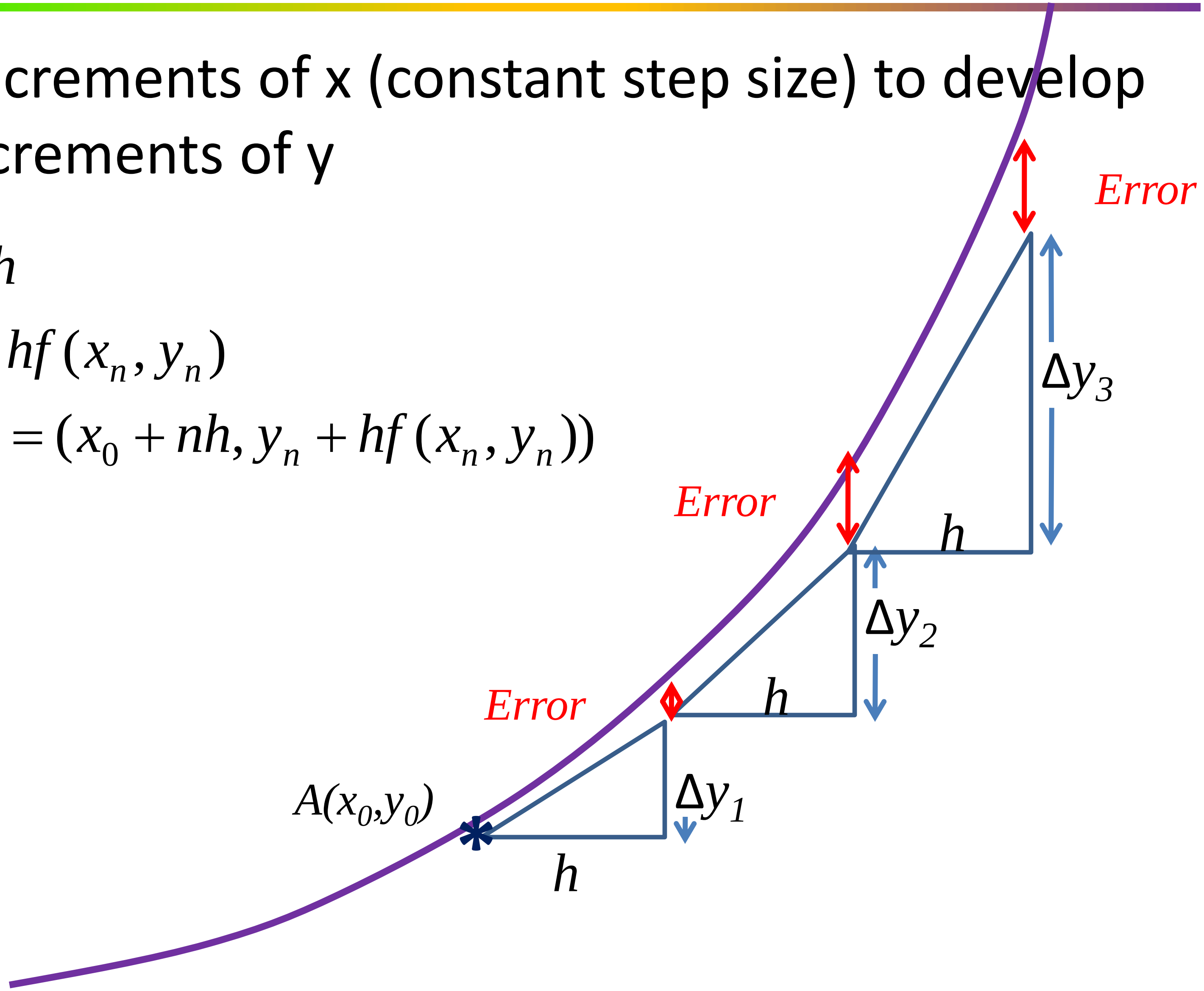
Euler's Method

We use increments of x (constant step size) to develop new increments of y

$$x_n = x_0 + nh$$

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$\therefore Z(x_n, y_n) = (x_0 + nh, y_n + hf(x_n, y_n))$$



Example using Euler' Method

Estimate the value at x=1 for the differential equation

$$\frac{dy}{dx} = x^2 + y^2$$

which has passes through the point (0,0.5), using a step length of 0.25

Using Columns and Rigour

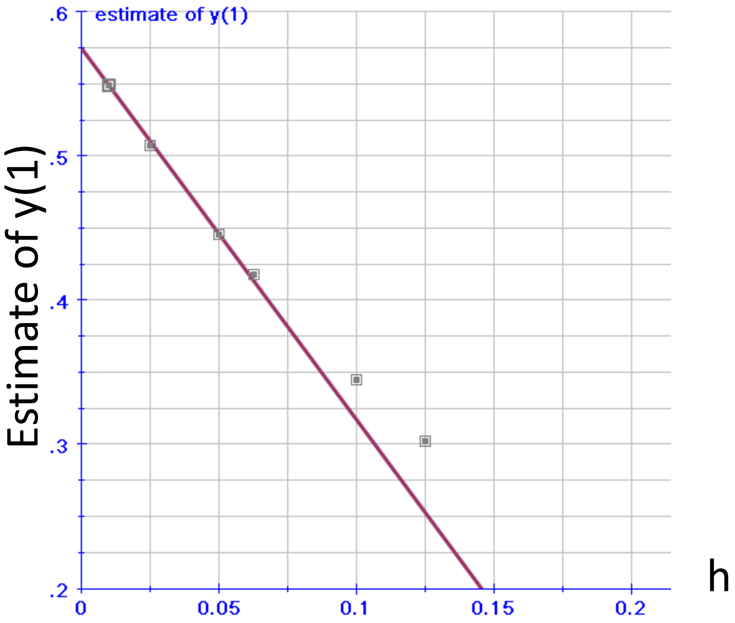
To get to x=1 from x=0, with step size 0.25

1	h=	0.25				
2	dy/dx=	x^2+y^2				
3	init x=	0	init y=	0.5		
4	n	x	y	dy/dy	h(dy/dx)	new y=y+h(dy/dx)
5	0	0.0000	0.5000000	0.2500	0.0625	0.5625
6	1	0.2500	0.5625000	0.3789	0.0947	0.6572
7	2	0.5000	0.6572266	0.6819	0.1705	0.8277
8	3	0.7500	0.8277133	1.2476	0.3119	1.1396
9	4	1.0000	1.1396156	2.2987	0.5747	1.7143

The estimate for y(1) will be 1.1396 (4dp)

How do we know when to stop?

h	Estimate for y(1)
0.2	1.1964
0.125	1.3020
0.1	1.3447
0.0625	1.4185
0.05	1.4463
0.025	1.5081
0.01	1.5498



Plotting the information from the table we can see that by drawing a straight line through the two points with the smallest step size, the other points are all approaching this line

Error estimate

h	Estimate for y(1)
0.025	1.5081
0.01	1.5498
0.001	1.5770
0.0001	1.5798

As the step size decreases, the estimates for y(1) appear to lie very close to a straight line

To get a solution correct to 2 decimal places using the spreadsheet, you need to find a step such that making the step smaller does not alter the first two decimal places.

We need somewhere between 1000 and 10,000 steps.

Estimating the exact value for y(1)

For small values of *h*, the error is approximately a linear function. Using $y=mh+c$ we need to find m and c

Using $h = 0.025$, $y(1) = 1.5080586$, let $y(1) = 0.025a + b$
and $h = 0.01$, $y(1) = 1.5498494$ and $y(1) = 0.01a + b$

$$1.5080586 = 0.025a + b \quad (1)$$
$$1.5498494 = 0.01a + b \quad (2)$$
$$(1) - (2) \quad \therefore -0.0417908 = 0.015a$$

$$\therefore a = -2.786053 \quad \& \quad b = 1.521988867$$

$$\Rightarrow \quad y(1) = 1.521988867 - 2.786053h$$

The exact value will be when $h = 0$

$$\therefore y(1) = 1.521988867$$